

Distances on a measure (affine ordered hovel)

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Abstract

A measure (a.k.a affine ordered hovel) \mathcal{I} is a generalization of the Bruhat-Tits building that is associated to a split Kac-Moody group G over a non-archimedean local field. This is a union of affine spaces called apartments. When G is a reductive group, \mathcal{I} is a building and there is a G -invariant distance inducing a norm on each apartment. In this paper, we study distances on \mathcal{I} inducing the affine topology on each apartment. We show that some properties (completeness, local compactness, ...) cannot be satisfied when G is not reductive. Nevertheless, we construct distances such that each element of G is a continuous automorphism of \mathcal{I} .

1 Introduction

If G is a split Kac-Moody group over a non-archimedean local field, Stéphane Gaussent and Guy Rousseau introduced a space \mathcal{I} on which G acts and they called this set a "measure" (or an "affine ordered hovel"), see [GR08], [Rou12]. We then consider G as a group of automorphisms of \mathcal{I} . This construction generalizes the construction of the Bruhat-Tits building associated to a split reductive group over a local field made by François Bruhat and Jacques Tits, see [BT72] and [BT84]. This measure is an object similar to a building. It is a union of subsets called "apartments", each one having a structure of affine space and an additional structure of apartment defined by hyperplanes (called walls) of this affine space. The group G acts by permuting the apartments and by inducing affine map on each apartment, sending walls on walls. We can also define sectors and retractions from \mathcal{I} onto apartments with center a sector, as in the case of Bruhat-Tits buildings. However there can be two points of \mathcal{I} which are not included in any apartment. Studying \mathcal{I} enables to get information on G and this is why we study measures.

Each Bruhat-Tits building B associated to a split reductive group H over a local field is equipped with a distance d such that H acts isometrically on B and such that the restriction of d to each apartment is a euclidean distance. These distances are important tools in the study of buildings. We will show that we cannot equip measures which are not buildings with distances having these properties but it seems natural to ask whether we can define distances on a measure having "good" properties. We limit our study to distances inducing the affine topology on each apartment. We show that under assumptions of continuity for retractions, the metric space we have is not complete nor locally compact (see Subsection 3.3). We show that there is no distance on \mathcal{I} such that the restriction to each apartment is a norm. However, we prove the following theorems (Corollary 5.8, Lemma 4.9, Corollary 5.9 and Theorem 5.14): let \mathfrak{q} be a sector germ of \mathcal{I} , then there exists a distance d on \mathcal{I} having the following properties:

- the topology induced on each apartment is the affine topology
- each retraction with center \mathfrak{q} is 1-Lipschitz continuous
- each retraction with center a sector-germ of the same sign as \mathfrak{q} is Lipschitz continuous
- each $g \in G$ is Lipschitz continuous when we see it as an automorphism of \mathcal{I} .

We call the distances constructed in the proof of this theorem distances of *positive* or of *negative type*, depending on the sign of \mathfrak{q} . A distance of positive or negative type is called a signed distance. We prove that all distances of positive type on a measure (resp. of negative type) are equivalent, where we say that two distances d_1 and d_2 are equivalent if there exist $k, l \in \mathbb{R}_+^*$ such that $kd_1 \leq d_2 \leq ld_1$ (this is Theorem 5.7). We thus get a *positive topology* \mathcal{T}_+ and a *negative topology* \mathcal{T}_- . We prove (Corollary 6.4) that these topologies are different when \mathcal{I} is not a building. When \mathcal{I} is a building these topologies are the usual topology on a building (Proposition 5.15).

Let \mathcal{I}_0 be the orbit of some special vertex under the action of G . If \mathcal{I} is not a building, \mathcal{I}_0 is not discrete for \mathcal{T}_- and \mathcal{T}_+ . We also prove that if ρ is a retraction centered at a positive (resp. negative) sector-germ, ρ is not continuous for \mathcal{T}_- (resp. \mathcal{T}_+), see Proposition 6.3. For this reasons we introduce *combined distances*, which are the sum of a distance of positive type and of a distance of negative type. We then have the following theorem (Theorem 6.7): all the combined distances on \mathcal{I} are equivalent; moreover, if d is a combined distance and \mathcal{I} is equipped with d we have:

- each $g : \mathcal{I} \rightarrow \mathcal{I} \in G$ is Lipschitz continuous
- each retraction centered at a sector-germ is Lipschitz continuous
- the topology induced on each apartment is the affine topology
- there exists $\delta > 0$ such that for all $x, x' \in \mathcal{I}_0$, $d(x, x') < \delta$ implies $x = x'$.

The topology \mathcal{T}_c associated to combined distances is the initial topology with respect to the retractions of \mathcal{I} (see Corollary 6.10).

Let us explain how we define distances of positive or negative type. Let A be an apartment and Q be a sector of A . Maybe considering $g.A$ for some g in G , one can suppose that $A = \mathbb{A}$, the standard apartment of \mathcal{I} and $Q = C_f^v$, the fundamental chamber of \mathbb{A} (or $Q = -C_f^v$ but this case is similar). Let N be a norm on \mathbb{A} . If $x \in \mathcal{I}$, there exists an apartment A_x containing x and $+\infty$ (which means that A_x contains a sub-sector of C_f^v). For $q \in \overline{C_f^v}$, we define $x + q$ as the translate of x by q in A_x . When q is made more and more dominant, $x + q \in C_f^v$. Therefore, for all $x, x' \in \mathcal{I}$, there exists $q, q' \in C_f^v$ such that $x + q = x' + q'$. We then define $d(x, x')$ to be the minimum of the $N(q) + N(q')$ for such couples q, q' .

We thus obtain a distance for each sector Q and for each norm N on an apartment containing Q . We show that this distance only depends on the germ of Q and on N (in Subsection 4.4).

In Section 2, we set the general frameworks and define measures.

In Section 3, we show that if \mathfrak{q} is a sector-germ of \mathcal{I} , we can write each apartment as a finite union of closed convex parts such that each part is included in an apartment containing \mathfrak{q} . The most important case for us is when A contains a sector-germ adjacent to \mathfrak{q} . We can then write A as the union of two half-apartments, each included in an apartment containing \mathfrak{q} . At the end of this part, we show restrictions we have on distances on a measure.

In Section 4, we construct signed distances on \mathcal{I} . In Section 5, we use the results of the first section to show that the distances of the same sign are equivalent. We use it to show the properties mentioned in the abstract. In Section 6, we first show that when \mathcal{I} is not a building, \mathcal{T}_+ and \mathcal{T}_- are different. Then we define combined distances and study their properties. In Section 7, we show that \mathcal{I} is contractible for the topologies we introduced.

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2 General frameworks

In this section, we describe measures we are going to study. They have the following properties: they are semi-discrete, thick of finite thickness and there is a group (vectorially Weyl) acting strongly transitively on them (see [Rou11] 1 and [GR14] 1 for definitions). These properties are satisfied by measures associated to (quasi) split Kac-Moody groups over local fields. We begin by defining the standard apartment. References for this Section are [Kac94], Chapter 1 and 3, [GR08] Section 2 and [GR14] Section 1.

2.1 Root generating system

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix $C = (c_{i,j})_{i,j \in I}$ with integers coefficients, indexed by a finite set I and satisfying:

1. $\forall i \in I, c_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j, c_{i,j} \leq 0$
3. $\forall (i,j) \in I^2, c_{i,j} = 0 \Leftrightarrow c_{j,i} = 0$.

A root generating system is a 5-tuple $\mathcal{S} = (C, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ made of a Kac-Moody matrix C indexed by I , of two dual free \mathbb{Z} -modules X (of characters) and Y (of cocharacters) of finite rank $\text{rk}(X)$, a family $(\alpha_i)_{i \in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i \in I}$ (of simple coroots) in Y . They have to satisfy the following compatibility condition: $c_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i \in I}$ is free in X and that the family $(\alpha_i^\vee)_{i \in I}$ is free in Y .

We now fix a Kac-Moody matrix C and a root generating system with matrix C .

Let $V = Y \otimes \mathbb{R}$. Every element of X induces a linear form on V . We will consider X as a subset of the dual V^* of V : the $\alpha_i, i \in I$ are viewed as linear form on V . For $i \in I$, we define an involution r_i of V by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in V$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\text{GL}(V)$ generated by the α_i for $i \in I$ is denoted by W^v and is called the *Weyl group* of \mathcal{S} .

Let $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$. We call Q^\vee the *coroot-lattice*.

One defines an action of the group W^v on V^* by the following way: if $x \in V, w \in W^v$ and $\alpha \in V^*$ then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$, be the set of *real roots*. Then $\Phi \subset Q$, where $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$. Let $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i, Q^- = -Q^+, \Phi^+ = \Phi \cap Q^+$ and $\Phi^- = \Phi \cap Q^-$. Then $\Phi = \Phi^+ \cup \Phi^-$. The element of Φ^+ (resp. Φ^-) are called the *real positive roots* (resp. *real negative roots*). Let $W^a = Q^\vee \rtimes W^v \subset \text{GA}(V)$ the *affine Weyl group* of \mathcal{S} , where $\text{GA}(V)$ is the group of affine isomorphisms of V .

A *wall* M (resp. a half-apartment D) of V is a hyperplane (resp. a half-space) of the shape $\alpha^{-1}\{k\}$ (resp. $\alpha^{-1}([k, +\infty))$) for some $k \in \mathbb{R}$. The wall M (resp. half-apartment) is said to be a *true wall* (resp. a true half-apartment) if $k \in \mathbb{Z}$ and a *ghost wall* if $k \notin \mathbb{Z}$.

2.2 Vectorial faces

Define $C_f^v = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$. We call it the *fundamental chamber*. For $J \subset I$, one sets $F^v(J) = \{v \in V \mid \alpha_i(v) = 0 \forall i \in J, \alpha_i(v) > 0 \forall i \in J \setminus I\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The *positive* (resp. *negative*) *vectorial faces* are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A *vectorial face* is either a positive vectorial face or a negative vectorial face. We call *positive chamber* (resp. *negative*) every cone of the shape $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). By Section 1.3 of [Rou11], the action of W^v on the positive chambers is simply transitive. The *Tits cone* \mathcal{T} is defined by $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. We also consider the negative cone $-\mathcal{T}$. We define a W^v -invariant relation \leq on V by: $\forall (x, y) \in V^2, x \leq y \Leftrightarrow y - x \in \mathcal{T}$.

2.3 Study of metric properties of W^v

In this subsection we prove that when W^v is infinite there exists no norm on V such that W^v is a group of isometries and we give a result of density of the walls of V .

Let $V_{in} = \bigcap_{i \in I} \ker \alpha_i = \bigcap_{w \in W^v} \ker(\text{Id} - w)$. Let $\overline{V} = V/V_{in}$. If $x \in V$, one denotes by \overline{x} its image in \overline{V} . Each $w \in W^v$ induces an automorphism \overline{w} of \overline{V} by the formula: $\overline{w}(\overline{x}) = \overline{w(x)}$ for all $x \in V$. Let $\overline{W^v} = \{\overline{w} \mid w \in W^v\} \subset \text{Aut}(\overline{V})$ where $\text{Aut}(\overline{V})$ is the group of linear automorphisms of \overline{V} . Let $\overline{Y} = \{\overline{y} \mid y \in Y\}$.

Lemma 2.1. 1. *The map $f : W^v \rightarrow \overline{W^v}$ defined by $f(w) = \overline{w}$ for all $w \in W^v$ is an isomorphism of groups.*

2. *The set \overline{Y} is a lattice of \overline{V} .*

Proof. Let $w \in W^v \setminus \{1\}$ and $x \in C^v$. As the action of W^v on the positive chambers is simply transitive, $w.x \notin C_f^v$. Therefore, there exists $i \in I$ such that $\alpha_i(w.x) < 0$. In particular $\alpha_i(w.x - x) < 0$ and thus $w.x - x \notin V_{in}$. As a consequence, $\overline{w} \neq 1$ and f is injective, which proves 1.

As Y spans V , \overline{Y} spans \overline{V} . Let $||$ be a norm on V . One equips \overline{V} with the quotient norm $|||$: for all $x \in V$, $||\overline{x}|| = \inf_{y \in \overline{x}} |y|$.

Let $(z_n) \in \overline{Y}^{\mathbb{N}}$ be a sequence converging towards 0. For all $n \in \mathbb{N}$, one writes $z_n = \overline{y_n}$ with $y_n \in Y$, one chooses $u_n \in \overline{y_n}$ such that $|u_n| \leq 2|z_n|$ and one writes $u_n = x_n + y_n$, with $x_n \in V_{in}$.

As V is finite dimensional, there exists $w_1, \dots, w_l \in W^v$ such that $V_{in} = \bigcap_{i=1}^l \ker(w_i - \text{Id})$. For all $i \in \llbracket 1, l \rrbracket$, $w_i.u_n = x_n + w_i.y_n \rightarrow 0$. Consequently $y_n - w_i.y_n \rightarrow 0$ for all $i \in \llbracket 1, l \rrbracket$. As $y_n - w_i.y_n \in Y$ for all $(i, n) \in \llbracket 1, l \rrbracket \times \mathbb{N}$, $y_n \in V_{in}$ for n large enough. Thus $z_n = \overline{y_n} = 0$ for n large enough, which shows that \overline{Y} is a lattice of \overline{V} . \square

Lemma 2.2. *Let Z be a finite dimensional vectorial space on \mathbb{R} and H be a subgroup of $\text{GL}(Z)$. Suppose that H is infinite and stabilizes a lattice L . Then for all norm on Z , H is not a group of isometries.*

Proof. Let (e_1, \dots, e_k) be a basis of Z such that $L = \mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_k$. Consider $f : H \rightarrow L^k$ defined by $f(h) = (h.e_1, \dots, h.e_k)$ for all $h \in H$. Then f is injective and thus there exists

$i \in \llbracket 1, k \rrbracket$ such that $\{h.e_i | h \in H\}$ is infinite. As $\{h.e_i | h \in H\} \subset L$, $\{h.e_i | h \in H\}$ is not bounded, which shows the lemma. \square

Let M be a true wall of V , $M = \beta^{-1}(\{k\})$, with $\beta \in \Phi$ and $k \in \mathbb{Z}$. If $k = 0$, one says that M is a *true wall direction*. Two true walls $M_1 = \beta^{-1}(\{k_1\})$ and $M_2 = \beta^{-1}(\{k_2\})$ (with $\beta \in \Phi$ and $k_1, k_2 \in \mathbb{Z}$) of the same directions are called *consecutive* if $|k_1 - k_2| = 1$.

Proposition 2.3. 1. Suppose that there exists a norm $|| \cdot ||$ on V such that W^a is a group of isometries. Then W^v is finite.

2. Let $|| \cdot ||$ be a norm on V , d be the induced distance on V and suppose that W^v is infinite. Then for all $\epsilon > 0$ there exists a true wall direction M_0 such that for all consecutive walls M_1 and M_2 of direction M_0 , $d(M_1, M_2) < \epsilon$.

Proof. Part 1 is an consequence of Lemma 2.2, applied with $Z = V$, $H = W^v$ and $L = Y$.

Suppose that W^v is infinite. Let $|| \cdot ||$ be a norm on V and suppose that there exists $\eta > 0$ such that for all consecutive walls M_1 and M_2 , $d(M_1, M_2) \geq \eta$. Let $\beta \in \Phi$ and $x, y \in V$ such that $\beta(x) \neq \beta(y)$. Then $|\frac{1}{\beta(y-x)}(y-x)| \geq \eta$ and thus $\beta(y-x) \leq \frac{|y-x|}{\eta}$. For all $x \in V$, one sets $N(x) = \sup_{\beta \in \Phi} |\beta(x)|$. Define $\overline{N} : \overline{V} \rightarrow \mathbb{R}_+$ by $\overline{N}(\overline{x}) = N(x)$ for all $x \in V$. Then \overline{N} is a norm on \overline{V} and \overline{W}^v is a group of isometries for this norm. By Lemma 2.1, and Lemma 2.2 applied with $Z = \overline{V}$, $H = \overline{W}^v$ and $L = \overline{Y}$, this is absurd. Thus the proposition is proved. \square

2.4 Filters

Definition 2.4. A filter in a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E , if $S, S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$ then $S \in F$.

If F is a filter in a set E , and E' is a subset of E , one says that F contains E' if every element of F contains E' . If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. By abuse of language, we will sometimes say that E' is a filter by identifying $F_{E'}$ and E' . If F is a filter in E , its closure \overline{F} (resp. its convex envelope) is the filter of subsets of E containing the closure (resp. the convex envelope) of some element of F . A filter F is said to be contained in an other filter F' : $F \subset F'$ (resp. in a subset Z in E : $F \subset Z$) if and only if any set in F' (resp. if Z) is in F .

A *sector* in V is a set of the shape $S = x + C^v$ with $C^v = \pm w.C_f^v$ for some $x \in V$ and $w \in W^v$. The point x is its *base point* and C^v is its *direction*. The intersection of two sectors of the same direction is a sector of the same direction.

The *sector-germ* of a sector $S = x + C^v$ is the filter \mathfrak{s} of subsets of V containing a V -translate of S . It only depends on the direction C^v . We denote by $+\infty$ (resp. $-\infty$) the sector-germ of C_f^v (resp. $-C_f^v$). A sector $S = x + C^v$ is said to be *positive* (resp. *negative*) if $C^v = w.C_f^v$ (resp. $C^v = -w.C_f^v$) for some $w \in W^v$. The sign extends to sector-germs.

A ray δ with base point x and containing $y \neq x$ (or the interval $[x, y] = [x, y] \setminus \{x\}$ or $[x, y]$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $y - x \in \pm \overset{\circ}{\mathcal{T}}$, the interior of $\pm \mathcal{T}$.

We now denote by \mathbb{A} the affine space V equipped with its faces, chimneys, ...

In the next subsection, we give the definition of a measure. This definition uses objects which we have not defined. This definitions are available in [Rou11] 1.7 and 1.10 and in [GR14] 1.4. For a first reading, one can just know the following facts about these objects:

1. An enclosure $\text{cl}_{\mathbb{A}}(F)$ is associated to each filter F of \mathbb{A} . This is a filter in \mathbb{A} containing the convex envelope of the closure of F .
2. A face or a chimney is a filter in V .
3. A sector is a chimney which is solid and splayed.
4. If a chimney is a sector, its germ as a chimney coincides with its germ as a sector.
5. The group W^a permutes the sectors, the enclosures, the faces and the chimneys of V .

2.5 Measure

We now denote by \mathbb{A} the affine space V equipped with its faces, chimneys, ...

An apartment of type \mathbb{A} is a set A with a nonempty set $\text{Isom}(\mathbb{A}, A)$ of bijections (called Weyl isomorphisms) such that if $f_0 \in \text{Isom}(\mathbb{A}, A)$ then $f \in \text{Isom}(\mathbb{A}, A)$ if and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. A Weyl isomorphism between two apartments $\phi : A \rightarrow A'$ is a bijection such that ($f \in \text{Isom}(\mathbb{A}, A)$ if and only if $\phi \circ f \in \text{Isom}(\mathbb{A}, A')$). We extend all the notions that are preserved by W^a to each apartment. By the fact 5 of the above subsection, sectors, enclosures, faces and chimneys are well defined in any apartment of type \mathbb{A} .

Definition 2.5. *An affine measure of type \mathbb{A} is a set \mathcal{I} endowed with a covering \mathcal{A} of subsets called apartments such that:*

(MA1) *Any $A \in \mathcal{A}$ admits a structure of an apartment of type \mathbb{A} .*

(MA2) *If F is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment A and if A' is another apartment containing F , then $A \cap A'$ contains the enclosure $\text{cl}_A(F)$ of F and there exists a Weyl isomorphism from A onto A' fixing $\text{cl}_A(F)$.*

(MA3) *If \mathfrak{R} is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains \mathfrak{R} and F .*

(MA4) *If two apartments A, A' contain \mathfrak{R} and F as in (MA3), then there exists a Weyl isomorphism from A to A' fixing $\text{cl}_A(\mathfrak{R} \cup F)$.*

(MAO) *If x, y are two points contained in two apartments A and A' , and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.*

In this definition, we say that an apartment contains a germ of a filter if it contains at least one element of this germ. We say that a map fixes a germ if it fixes at least one element of this germ. In the sequel, we will say "isomorphism" instead of "Weyl isomorphism".

Each apartment A can be equipped with a structure of affine space by using an isomorphism of apartments $\phi : \mathbb{A} \rightarrow A$. We say that we fix a vectorial structure on A when we fix the origin of A at $\phi(0)$.

We assume that \mathcal{I} has a strongly transitive group of automorphisms G , which means that all isomorphisms involved in the above axioms are induced by elements of G . We choose in \mathcal{I} a fundamental apartment, that we identify with \mathbb{A} . As G is strongly transitive, the apartments of \mathcal{I} are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of \mathbb{A} induces a group $\nu(N)$ of affine automorphisms of \mathbb{A} and we suppose that $\nu(N) = W^v \ltimes Y$.

We suppose that \mathcal{I} is thick of finite thickness, see [GR14] 1.5 for a definition. This definition coincides with the usual one when \mathcal{I} is a building. We will not use this definition directly but we will use it to apply results of [Rou11] and finiteness results of [Héb16].

An example of such a measure \mathcal{I} is the measure associated to a split Kac-Moody group over an ultrametric field constructed in [GR08] and in [Rou12].

The measure \mathcal{I} is a building if and only if W^v is finite, see [Rou11] 2.2 6).

2.6 Retractions centered at sector-germs

Let \mathfrak{s} be a sector-germ of \mathcal{I} . Let $x \in \mathcal{I}$. By (MA3), there exists an apartment A_x of \mathcal{I} containing x and \mathfrak{s} . By (MA4), there exists an isomorphism of apartments $\phi : A_x \rightarrow A$ fixing \mathfrak{s} . By [Rou11] 2.6, $\phi(x)$ does not depend on the choices we made and thus we can let $\rho_{A,\mathfrak{s}}(x) = \phi(x)$.

The map $\rho_{A,\mathfrak{s}}$ is a retraction from \mathcal{I} onto A . It only depends on \mathfrak{s} and A and we call it the *retraction onto A centered at \mathfrak{s}* .

If A and B are two apartments, and $\phi : A \rightarrow B$ is an isomorphism of apartments fixing some set X , one writes $\phi : A \xrightarrow{X} B$. If A and B share a sector-germ \mathfrak{q} , one denotes by $A \xrightarrow{A \cap B} B$ or by $A \xrightarrow{\mathfrak{q}} B$ the unique isomorphism of apartments from A to B fixing \mathfrak{q} (and also $A \cap B$). We denote by $\mathcal{I} \xrightarrow{\mathfrak{q}} A$ the retraction onto A fixing \mathfrak{q} .

2.7 Parallelism in \mathcal{I} and definition of y_ν and T_ν

Let us explain briefly the notion of parallelism in \mathcal{I} . This is done more completely in [Rou11] Section 3.

Let us begin with rays. Let δ and δ' be two generic rays in \mathcal{I} . By (MA3) and [Rou11] 2.2 3) there exists an apartment A containing sub-rays of δ and δ' and we say that δ and δ' are *parallel*, if these sub-rays are parallel in A . Parallelism is an equivalence relation and its equivalence classes are called *directions*. Let Q be a sector of \mathcal{I} and A be an apartment containing Q . One fixes the origin of A in the base point of Q . Let $\nu \in Q$ and $\delta = \mathbb{R}_+\nu$. Then δ is a generic ray in \mathcal{I} . By Lemma 3.2 of [Héb16], for all $x \in \mathcal{I}$, there exists a unique ray $x + \delta$ of direction δ and base point x . To obtain this ray, one can choose an apartment A_x containing x and a sub-ray δ' of δ , which is possible by (MA3) and [Rou11] 2.2 3), and then we take the translate of δ' in A_x having x as a base point.

Let us recall briefly results from "Definition of y_ν and T_ν " of Section 3 of [Héb16]. Let $x \in \mathcal{I}$, one has $x + \delta \cap A = y_\nu(x) + \delta$ for some $y_\nu(x) \in A$. Let \mathfrak{q} be the germ of Q and $\rho : \mathcal{I} \xrightarrow{\mathfrak{q}} A$. One has $y_\nu(x) = \rho(x) + T_\nu(x)\nu$, for some $T_\nu(x) \in \mathbb{R}_+$. This defines two maps $y_\nu : \mathcal{I} \rightarrow A$ and $T_\nu : \mathcal{I} \rightarrow \mathbb{R}_+$ and we have $y_\nu(x) = x$ and $T_\nu(x) = 0$ for all $x \in A$. For all $x \in \mathcal{I} \setminus A$, $y_\nu(x) \neq x$ and $T_\nu(x) > 0$.

A *sector-face* f of \mathbb{A} , is a set of the shape $x + F^v$ for some vectorial face F^v and some $x \in \mathbb{A}$. The germ $\mathfrak{F} = \text{germ}_\infty(f)$ of this sector face is the filter containing the elements of the shape $q + f$, for some $q \in \overline{F^v}$. The sector-face f is said to be *spherical* if $F^v \cap \mathring{\mathcal{T}}$ is nonempty. A *sector-panel* is a sector-face included in a true wall and spanning this one as an affine space. A sector-panel is *spherical* (see [Rou11] 1). We extend these notions to \mathcal{I} thanks to the isomorphisms of apartments. Let us make a summary of the notion of parallelism for sector-faces. This is also more complete in [Rou11], 3.3.4)).

If f and f' are two spherical sector-face, there exists an apartment B containing their germs \mathfrak{F} and \mathfrak{F}' . One says that f and f' are *parallel* if $\mathfrak{F} = \text{germ}_\infty(x + F^v)$ and $\mathfrak{F}' = \text{germ}_\infty(y + F^v)$ for some $x, y \in B$ and for some vectorial face F^v of B . Parallelism is an equivalence relation. The parallelism class of a sector-face germ \mathfrak{F} is denoted \mathfrak{F}_∞ . If M is a wall of \mathcal{I} , its direction M^∞ is the set of \mathfrak{F}_∞ such that $\mathfrak{F} = \text{germ}_\infty(f)$, with f a spherical sector-face included in M . We denote by \mathcal{I}^∞ the set of directions of spherical faces of \mathcal{I} . The sector-faces of a wall M^∞ of \mathcal{I}^∞ are the elements of this wall.

By Proposition 4.7.1) of [Rou11], for all $x \in \mathcal{I}$ and all $\mathfrak{F}_\infty \in \mathcal{I}^\infty$, there exists a unique sector-face $x + \mathfrak{F}_\infty$ of direction \mathfrak{F}_∞ and with base point x . The existence can be obtained in the same way as for rays.

Let \mathfrak{q} be a sector-germ of \mathcal{I} and $x \in \mathcal{I}$. Let A be an apartment containing x and \mathfrak{q} (such an apartment exists by (MA3)). Let $x + \overline{\mathfrak{q}}_A$ be the closure of $x + \mathfrak{q}$ in A (for the topology induced by its structure of affine space). Then $x + \overline{\mathfrak{q}}_A$ does not depend on the choice of A containing x and \mathfrak{q} : let B be an apartment containing x and \mathfrak{q} and $\phi : A \xrightarrow{A \cap B} B$. Then $A \cap B = \{x \in A \mid \phi(x) = x\}$. By (MA4), ϕ fixes the enclosure of $x + \mathfrak{q}$, which contains $x + \overline{\mathfrak{q}}_A$. Therefore $x + \overline{\mathfrak{q}}_B \supset x + \overline{\mathfrak{q}}_A$ and by symmetry, $x + \overline{\mathfrak{q}}_A = x + \overline{\mathfrak{q}}_B$. We denote this set by $x + \overline{\mathfrak{q}}$.

Let f, f' be sector-faces. One says that f dominates f' (resp. f and f' are opposite) if one can write $\text{germ}_\infty(f) = \text{germ}_\infty(x + \mathfrak{F})$, $\text{germ}_\infty(f') = \text{germ}_\infty(x' + F'^v)$ for some $x, x' \in \mathcal{I}$ and F^v, F'^v two vectorial faces of a same apartment of \mathcal{I} such that $\overline{F}^v \supset F'^v$ (resp. such that $F'^v = -F^v$). By Proposition 3.2 2) and 3) of [Rou11], these notions extend to \mathcal{I}^∞ .

3 Splitting of apartments

3.1 Splitting of apartments in two half in two half-apartment

The aim of this section is to show that if A is an apartment, M is a true wall of A , \mathfrak{F} is a sector-panel of M^∞ and \mathfrak{q} is a sector-germ dominating \mathfrak{F}_∞ , there exist two opposite half-apartments D_1 and D_2 of A such that their wall is parallel to M and such that for all $i \in \{1, 2\}$, D_i and \mathfrak{q} are included in some apartment, which is Lemma 3.6. This section will enable us to show that for each choice of sign, the distances we construct are equivalent.

The next lemma enables to prove that if the intersection of two apartment is a half-apartment, it is a true half-apartment. In the proof of this lemma, we use the notion of Hecke paths but we do not define it precisely and we will not use it in the sequel of this paper. Hecke paths are more or less images by retractions of ordered segments in \mathcal{I} . They are defined in Section 1 of [GR14]. A Hecke path in an apartment A is a piecewise linear function $\pi : [0, 1] \rightarrow A$ satisfying some conditions. The function π is differentiable on $[0, 1]$ except maybe in a finite number of points and is left differentiable and right differentiable at each point where it makes sense. We will use the fact that if π is not differentiable in $t \in [0, 1]$, then $\pi(t)$ is in some true wall.

Lemma 3.1. *Let A be an apartment of \mathcal{I} on which we fix a structure of vectorial space, Q be a sector based at 0 and $\nu \in Q$. Let y_ν be as in Subsection 2.7. Let $x \in \mathcal{I} \setminus A$. Then $y_\nu(x)$ is in some true wall of A .*

Proof. Let B be an apartment containing x and \mathfrak{q} and $\phi : B \xrightarrow{\mathfrak{q}} A$. Let $\tau : [0, 1] \rightarrow B$ defined by $\tau(t) = \phi^{-1}(\phi(x) + 2tT_\nu(x)\nu)$. Then τ is a segment of B such that $\tau(0) = x$ and $\tau(1) = y_\nu(x) + T_\nu(x)\nu$. By Lemma 3.5 a) of [Héb16], $x \leq y_\nu(x)$. As $y_\nu(x) \leq y_\nu(x) + T_\nu(x)\nu$, $\tau(0) \leq \tau(1)$. Let $-Q$ be the sector of A opposite to Q and $-\mathfrak{q}$ be its germ. Let $\rho_- : \mathcal{I} \xrightarrow{-\mathfrak{q}} A$. By paragraph 2.3 of [GR14], $\pi = \rho_- \circ \tau$ is a Hecke path of shape $2T_\nu(x)$ with respect to $-Q$. For $t \in [\frac{1}{2}, 1]$, $\tau(t) = \pi(t)$ because $\tau(t) \in A$. Therefore $\pi'_+(\frac{1}{2}) = 2T_\nu(x)\nu$. If $t \in [0, \frac{1}{2})$, $\tau(t) \notin A$ (by definition of $T_\nu(x)$) and thus Lemma 3.6 of [Héb16] applied to $\tilde{\tau} : t \mapsto \tau(2t)$ shows that $\pi'_-(\frac{1}{2}) \neq 2T_\nu(x)\nu$. By (iii) of definition of Hecke paths in [GR14], there exists a real root β such that $\beta(\pi(\frac{1}{2})) = \beta(y_\nu(x)) \in \mathbb{Z}$. \square

Lemma 3.2. *Let A, B be two distinct apartments of \mathcal{I} containing a half-apartment D . Then $A \cap B$ is a true half-apartment.*

Proof. Let us first prove that $A \cap B$ is a half-apartment. Let M be the wall of D . We fix an origin of A in M . Let N be a supplementary of M in A included in the tits cone $0 \pm \mathcal{T}$, u be

such that $N = \mathbb{R}u$ and such that $D \cap N = \mathbb{R}^-u$. Let $n : A \rightarrow \mathbb{R}$ be such that $n(m + \lambda u) = \lambda$ for all $m \in M$ and $\lambda \in \mathbb{R}$. Let $X = \{n(x) | x \in A \text{ and } x + D \subset A \cap B\}$.

Let $\lambda = \sup X$ and $(\lambda_n) \in X^{\mathbb{N}}$ be such that $\lambda_n \rightarrow \lambda$. For all $n \in \mathbb{N}$, $\lambda_n u + D \subset A \cap B$ and as $A \neq B$, $\lambda < +\infty$. By (MA2), $A \cap B$ contains the enclosure of $\text{germ}_x([x, x_0] \setminus \{x\})$, where $x = \lambda u$ and $x_0 = \lambda_0 u$ and thus $A \cap B \ni x$. Let $y \in A \setminus (x + D)$ and suppose $y \in B$. Let \mathfrak{F} and \mathfrak{F}' be two opposite wall-sector of M and \mathfrak{q}' be the sector-germ of \mathcal{I} dominating \mathfrak{F}' and included in D . Then B contains $y + \mathfrak{F}$ and thus $\bigcup_{z \in y + \mathfrak{F}} z + \mathfrak{q}' = y + D$, which is absurd. Consequently, $A \cap B = x + D$: $A \cap B$ is a half-apartment.

Let F be the wall of $x + D$. It remains to show that F is true. Let $Z = \{z \in F | z \text{ is not included in any true wall except maybe } F\}$. Let μ be a Lebesgue measure on F . As the set of true walls of A is countable, $\mu(F \setminus Z) = 0$ and thus Z is not empty. Let $z \in Z$. Let Q be a sector of $A \cap B$ having z as a base point. One fixes the origin of A in z . Let $\nu \in Q$ and $y_\nu : \mathcal{I} \rightarrow A$ be as in the paragraph "Definition of y_ν and T_ν ". Let $-\nu$ be the opposite of ν in A . Let $\phi : A \xrightarrow{A \cap B} B$ and $x = \phi^{-1}(-\nu)$. Then $x \notin A$ and $y_\nu(x) = z$. By Lemma 3.1, z is in some true wall M and by choice of z we deduce that $M = F$. \square

Lemma 3.3. *Let M be a true wall of \mathbb{A} and $\phi \in W^a$ be an element fixing M . Then $\phi \in \{\text{Id}, s\}$, where s is the reflection of W^a with respect to M .*

Proof. One writes $w = \tau \circ u$, with $u \in W^v$ and τ a translation of \mathbb{A} . Then $u(M)$ is a wall parallel to M . Let M_0 be the wall parallel to M containing 0. Then $u(M_0)$ is a wall parallel to M_0 and containing 0: $u(M_0) = M_0$. Let C be a vectorial chamber adjacent to M_0 . Then $u(C)$ is a chamber adjacent to C : $u(C) \in \{C, s_0(C)\}$, where s_0 is the reflection of W with respect to M_0 . Maybe composing u by s_0 , one can suppose that $u(C) = C$ and thus $u = \text{Id}$ (because the action of W^v on the set of chambers is simply transitive). \square

If D is a half apartment of \mathcal{I} , one sets \mathfrak{D} , the filter of half apartment containing a shortening of D . If D_1 and D_2 are two half-apartments, one says that $D_1 \sim D_2$ if $\mathfrak{D}_1 = \mathfrak{D}_2$ and one says that D_1 and D_2 have opposite direction if there exists an apartment A containing a shortening of them, if their walls are parallel, and if D_1 and D_2 are not equivalent. One says D_1 and D_2 are opposite if they have opposite directions and if $D_1 \cap D_2$ is a wall.

Lemma 3.4. *Let A_1, A_2, A_3 be distinct apartments. Suppose that $A_1 \cap A_2, A_1 \cap A_3$ and $A_2 \cap A_3$ are half-apartments such that $A_1 \cap A_3$ and $A_2 \cap A_3$ have opposite directions. Then $A_1 \cap A_2 \cap A_3 = M$ where M is the wall of $A_1 \cap A_3$, and for all $(i, j, k) \in \{1, 2, 3\}^3$ such that $\{i, j, k\} = \{1, 2, 3\}$, $A_i \cap A_j$ and $A_i \cap A_k$ are opposite. Moreover, if $s : A_3 \rightarrow A_3$ is the reflection with respect to M , $\phi_1 : A_3 \xrightarrow{A_1 \cap A_3} A_1$, $\phi_2 : A_3 \xrightarrow{A_2 \cap A_3} A_2$ and $\phi_3 : A_1 \xrightarrow{A_2 \cap A_1} A_2$, then the following diagram is commutative:*

$$\begin{array}{ccc} A_3 & \xrightarrow{s} & A_3 \\ \downarrow \phi_2 & & \downarrow \phi_1 \\ A_2 & \xrightarrow{\phi_3} & A_1 \end{array}$$

Proof. By "Propriété du Y" (Section 4.9 of [Rou11]), $A \cap A_1 \cap A_2$ is nonempty. Let $x \in A \cap A_1 \cap A_2$. Let N be the wall parallel to the wall of $A_1 \cap A_3$ containing x . Then $A_1 \cap A_2 \cap A_3 \supset N$. Let $D_1 = A_1 \cap A_3$ and $D_2 = A_2 \cap A_3$. Then D_1 and D_2 have opposite directions and are not disjoint: they contain M and thus $A_1 \cap A_2 \cap A_3 \supset M$. Let us show that $\phi_2^{-1} \circ \phi_3 \circ \phi_1 = s$. The half-apartments $D_3 = A_1 \cap A_2$ and $D_1 = A_1 \cap A_3$ contains M and thus their walls are parallel. Suppose D_3 and D_2 are equivalent. Then $D_3 \supset \mathfrak{D}_2$ and $D_1 \cap D_2 = A_1 \cap A_2 \cap A_3 = D_3 \cap D_2 \supset \mathfrak{D}_2$. Therefore, D_1 and D_2 are equivalent, which

is absurd. The half-apartments $\phi_1(D_1) = D_1$ and $\phi_1(D_2)$ have opposite directions in A_1 , hence $\phi_1(D_2) \sim D_3$. Consequently $\phi_3 \circ \phi_1(D_2) \sim \phi_3(D_3) = D_3$. We also have $\phi_2(D_2) = D_2$. Therefore, $\phi_2^{-1} \circ \phi_3 \circ \phi_1 \neq \text{Id}$. By Lemma 3.3, $\phi_2^{-1} \circ \phi_3 \circ \phi_1 = s$. As s fixes $A_1 \cap A_2 \cap A_3$, $A_1 \cap A_2 \cap A_3 \subset M$ and thus $A_1 \cap A_2 \cap A_3 = M$. \square

Lemma 3.5. *Let M^∞ be a wall of \mathcal{I}^∞ . Let \mathfrak{q} be a sector-germ dominating a sector-panel of direction $\mathfrak{F}^\infty \subset M^\infty$. Let A_1 (resp. A_2) be an apartment containing a wall M_1 (resp. M_2) of direction M^∞ and \mathfrak{q} . Then either $A_1 = A_2$ or $A_1 \cap A_2$ is a half-apartment.*

Proof. Let Q be a sector of germ \mathfrak{q} and included in $A_1 \cap A_2$. Let N_1 be a wall parallel to M_1 and such that $N_1 \cap Q \neq \emptyset$. Let $\phi : A_1 \xrightarrow{A_1 \cap A_2} A_2$ and $N_2 = \phi(N_1)$. Let $f \subset N_1 \cap B$ be a sector-panel of direction \mathfrak{F}^∞ . Then $N_2 \supset f$. Let H be a wall of A_2 parallel to M_2 such that $H \supset f$. Then H is the affine space of A_2 spanned by f and N_2 too: $H = N_2$. Therefore, $N_2^\infty = M^\infty = N_1^\infty$. By Proposition 4.8 2) of [Rou11] $N_1 = N_2$ (this is the unique wall of direction M^∞ containing $\mathfrak{F} = \text{germ}_\infty(f)$). Let \mathfrak{F}' be the sector-panel germ of N_1 opposite to $\mathfrak{F} = \text{germ}_\infty(f)$. Then by (MA4), $A_1 \cap A_2 \supset \text{cl}(\mathfrak{F}' \cup \mathfrak{q})$, which is a half apartment and one concludes with Lemma 3.2. \square

The following lemma is similar to Proposition 2.9.1) of [Rou11].

Lemma 3.6. *Let A be an apartment, M be a wall of A and M^∞ be its direction. Let \mathfrak{F}_∞ be the direction of a sector-panel of M^∞ and \mathfrak{q} be a sector-germ dominating \mathfrak{F}_∞ and not included in A . Then there exists a unique pair $\{D_1, D_2\}$ of half-apartments of A such that:*

- D_1 and D_2 are opposite with common wall N parallel to M
- for all $i \in \{1, 2\}$, D_i and \mathfrak{q} are in some apartment A_i .

Moreover:

- D_1 and D_2 are true half-apartments
- such apartments A_1 and A_2 are unique and if D is the half-apartment of A_1 opposite to D_1 , then $D \cap D_2 = D_1 \cap D_2$ is a wall and $A_2 = D_2 \cup D$.

Proof. Let us first show the existence of D_1 and D_2 . Let \mathfrak{F}'_∞ be the sector-panel of M^∞ opposite to \mathfrak{F}_∞ . Let \mathfrak{q}'_1 and \mathfrak{q}'_2 be the sector-germ of A containing \mathfrak{F}'_∞ . For $i \in \{1, 2\}$, let A_i be an apartment of \mathcal{I} containing \mathfrak{q}'_i and \mathfrak{q} . Let $i \in \{1, 2\}$ and $x \in A \cap A_i$. Then $x + \mathfrak{q}'_i \subset A \cap A_i$ and $E_i = \bigcup_{y \in x + \mathfrak{q}'_i} y + \mathfrak{F}_\infty \subset A \cap A_i$ is a half-apartment of A and A_i .

Suppose $A_1 = A_2$. Then $A_1 \supset \bigcup_{x \in E_1} x + \mathfrak{q}'_2 = A$ and thus $A_1 = A \supset \mathfrak{q}$, which is absurd. For $i \in \{1, 2\}$, E_i and thus A_i contains a wall of direction M^∞ . By Lemma 3.5, $A_1 \cap A_2$ is a half-apartment.

By Lemma 3.4, $A_1 \cap A_2 \cap A = N$, where N is a wall of A parallel to M , and if $D_i = A \cap A_i$ for all $i \in \{1, 2\}$, $\{D_1, D_2\}$ fulfills the condition of the lemma. Let E_1, E_2 be another pair of opposite half apartments of A such that for all $i \in \{1, 2\}$, E_i and \mathfrak{q} are included in some apartment B_i and such that $E_1 \cap E_2$ is parallel to M . One can suppose $E_1 \sim D_1$ and $E_2 \sim D_2$. Suppose for example that $E_1 \subset D_1$. By the same reasoning as in the proof of the existence of D_1 and D_2 , $B_1 \cap A_2$ is a half apartment. Thus by Lemma 3.4, $B_1 \cap A_2 \cap A$ is a wall of A . But $B_1 \cap A_2 \cap A = E_1 \cap D_2 \subset D_1 \cap D_2 = N$, thus $E_1 \cap D_2 = N$ and hence $E_1 \supset D_1$. Therefore, $E_1 = D_1$ and by symmetry, $E_2 = D_2$. This proves the uniqueness of such a pair. The fact that D_1 and D_2 are true half-apartments comes from Lemma 3.2. Let us show the uniqueness of such apartments A_1 and A_2 . Let C_1 be an apartment containing D_1 and \mathfrak{q} . Let \mathfrak{F} be the sector-panel germ of N dominated by \mathfrak{q} . Let \mathfrak{F}' be the sector-panel germ of N opposite to \mathfrak{F} . Then by (MA4), $C_1 \supset \text{cl}(\mathfrak{F}', \mathfrak{q}) = D$ and thus $C_1 \supset A_1$. Therefore, $A_1 = C_1$. By symmetry, we get the lemma. \square

3.2 Splitting of apartments

In this subsection we mainly generalize Lemma 3.6. We show that if \mathfrak{q} is a sector germ of \mathcal{I} and if A is an apartment of \mathcal{I} , then A is the union of a finite number of convex closed parts P_i of A such that for all i , P_i and \mathfrak{q} are included in some apartment. This is Proposition 3.7. We then deduce a property of retractions with respect to a sector-germ. This subsection will not be used in the study of the distances we construct.

From now on, unless otherwise stated, "half-apartment" will implicitly refer to "true half-apartment".

Let \mathfrak{q} be a sector-germ and $\epsilon \in \{+, -\}$ be its sign. Let A be an apartment of \mathcal{I} . Then one sets $d_{\mathfrak{q}}(A) = \min\{d(\mathfrak{q}, \mathfrak{q}') \mid \mathfrak{q}' \text{ is a sector germ of } A \text{ of sign } \epsilon\}$. Let \mathcal{D}_A be the set of half-apartments of A . One sets $\mathcal{P}_{A,0} = \{A\}$ and for all $n \in \mathbb{N}^*$, $\mathcal{P}_{A,n} = \{\bigcap_{i=1}^n D_i \mid (D_i) \in (\mathcal{D}_A)^n\}$. The following proposition is very similar to Proposition 4.3.1 of [Cha10].

Proposition 3.7. *Let A be an apartment of \mathcal{I} , \mathfrak{q} be a sector-germ of \mathcal{I} et $n = d_{\mathfrak{q}}(A)$. Then there exist $P_1, \dots, P_k \in \mathcal{P}_{A,n}$, with $k \leq 2^n$ such that $A = \bigcup_{i=1}^k P_i$ and for each $i \in \llbracket 1, k \rrbracket$, P_i and \mathfrak{q} are contained in some apartment A_i . Moreover, for all $i \in \llbracket 1, k \rrbracket$, there exists an isomorphism $\psi_i : A_i \xrightarrow{P_i} A$.*

Proof. We do it by induction on n . This is clear if $n = 0$. Suppose this is true for all apartment B such that $d_{\mathfrak{q}}(B) \leq n - 1$. Let \mathfrak{s} be a sector-germ of A such that there exists a minimal gallery $\mathfrak{s} = \mathfrak{q}_1, \dots, \mathfrak{q}_n = \mathfrak{q}$ from \mathfrak{q} to \mathfrak{s} . Let M be a wall of A containing a sector-panel \mathfrak{F} dominated by \mathfrak{q}_1 and \mathfrak{q}_2 . Let D_1, D_2 be a pair of opposite half-apartments of wall parallel to M and such that for all $i \in \{1, 2\}$, D_i, \mathfrak{q}_2 is included in an apartment B_i (such a pair exists by Lemma 3.6). Let $i \in \{1, 2\}$. One has $d_{\mathfrak{q}}(B_i) = n - 1$ and thus $B_i = \bigcup_{j=1}^{k_i} P_j^{(i)}$, with $k_i \leq 2^{n-1}$, for all $j \in \llbracket 1, k_i \rrbracket$, $P_j^{(i)} \in \mathcal{D}_{B_i, n-1}$ and $\mathfrak{q}, P_j^{(i)}$ is contained in some apartment $A_j^{(i)}$. One has

$$A = D_1 \cap B_1 \cup D_2 \cap B_2 = \bigcup_{i \in \{1, 2\}, j \in \llbracket 1, k_i \rrbracket} P_j^{(i)} \cap D_i.$$

Let $i \in \{1, 2\}$ and $\phi_i : B_i \xrightarrow{A \cap B_i} A$. Let $j \in \llbracket 1, k_i \rrbracket$. Then we still have $P_j^{(i)} \cap D_i \subset A_j^{(i)}$. One writes $P_j^{(i)} = \bigcap_{l=1}^{n-1} E_l$ with $(E_l) \in \mathcal{D}_{B_i}^{n-1}$. We have $P_j^{(i)} \cap D_i = \phi_i(P_j^{(i)} \cap D_i) = \phi_i(P_j^{(i)}) \cap D_i$, and thus

$$P_j^{(i)} \cap D_i = D_i \cap \bigcap_{l=1}^{n-1} \phi_i(E_l) \in \mathcal{P}_{A,n}.$$

This shows the first part of this lemma. Let $i \in \{1, 2\}$ and $j \in \llbracket 1, k_i \rrbracket$. Let $f : A_i^{(j)} \xrightarrow{P_j^{(j)}} B_i$. Let $\psi = \phi_i \circ f$. One has $\psi|_{P_j^{(i)} \cap D_i} = \phi_i \circ \text{Id}_{P_j^{(i)} \cap D_i} = \text{Id}_{P_j^{(i)} \cap D_i}$. Thus $\psi : A \xrightarrow{P_j^{(i)} \cap D_i} A$, which completes the proof. \square

If A is an apartment and $x, y \in A$, one denotes by $[x, y]_A$ the segment joining x and y in A .

We deduce from the previous proposition a corollary which was already known for hovels associated to split Kac-Moody groups over ultrametric fields by Section 4.4 of [GR08]:

Corollary 3.8. *Let \mathfrak{q} be a sector-germ, A be an apartment and $x, y \in A$. Then there exists $x = x_1, \dots, x_k = y \in A$ such that $[x, y]_A = \bigcup_{i=1}^{k-1} [x_i, x_{i+1}]_A$ and such that for all $i \in \llbracket 1, k-1 \rrbracket$, $[x_i, x_{i+1}]_A$ and \mathfrak{q} are included in some apartment.*

Corollary 3.9. *Let A, B be two apartments of \mathcal{I} , \mathfrak{q} be a sector-germ of A and $\rho : B \rightarrow A$ be the retraction on A with center \mathfrak{q} . We choose norms on A and B . Then if a subset X of B is of nonempty interior, $\rho(X)$ is of nonempty interior.*

Proof. Let $n = d_{\mathfrak{q}}(B)$. One chooses $k \leq 2^n$ and $P_1, \dots, P_k \in \mathcal{P}_{B,n}$ such that $B = \bigcup_{i=1}^k P_i$ and such that for all $i \in \llbracket 1, k \rrbracket$, P_i, \mathfrak{q} are in some apartment A_i . For all $i \in \llbracket 1, k \rrbracket$, one chooses $\psi_i : A_i \xrightarrow{P_i} B$. Let $X \subset B$ be a set with nonempty interior. Then there exists $i \in \llbracket 1, k \rrbracket$ such that $P_i \cap X$ is of nonempty interior in B . Then $X \cap P_i = \psi_i(P_i \cap X)$ is of nonempty interior in A_i . Let $\phi : A_i \xrightarrow{\mathfrak{q}} A$. Then $\rho(X \cap P_i) = \phi(P_i \cap X)$ is of nonempty interior. \square

3.3 Restrictions on the distances

In this section, we show that some properties cannot be satisfied by distances on mesures. If A is an apartment of \mathcal{I} , we show that there exist apartments branching at all wall of A (this is Lemma 3.10). This implies that if \mathcal{I} is not a building the interior of each apartments is empty for the distances we study. We write \mathcal{I} as a countable union of apartment and then use Baire's Theorem to show that under rather weak assumption of regularity for retractions, a measure cannot be complete or locally compact for the distances we study.

Let us show a slight refinement of Corollaire 2.10 of [Rou11]:

Lemma 3.10. *Let A be an apartment of \mathcal{I} and D be a half-apartment of A . Then there exists an apartment B such that $A \cap B = D$.*

Proof. By Corollaire 2.10 of [Rou11], D is the intersection of two apartments, say A_1 and A_2 . If $A_1 = A$ or $A_2 = A$, there is nothing to prove. Suppose $A_1, A_2 \neq A$. Suppose $A_1 \cap A \neq D$ and $A_2 \cap A \neq D$. Then by Lemma 3.2, for all $i \in \{1, 2\}$, $A_i \cap A = D_i$, where D_i is a half apartment of A containing strictly D . But then $A_1 \cap A_2 \supset D_1 \cap D_2 \supsetneq D$, which is absurd. Therefore $A_1 \cap A = D$ or $A_2 \cap A = D$. \square

Proposition 3.11. *Suppose that there exists a distance $d_{\mathcal{I}}$ on \mathcal{I} such that for all apartment A , $d_{\mathcal{I}|A^2}$ is induced by some norm. Then \mathcal{I} is a building and $d_{\mathcal{I}|A^2}$ is W^a -invariant.*

Proof. Let \mathfrak{q} be a sector germ and A, B be two apartments containing \mathfrak{q} . Let $\phi : A \xrightarrow{\mathfrak{q}} B$. Let us show that $\phi : (A, d_{\mathcal{I}}) \rightarrow (B, d_{\mathcal{I}})$ is an isometry. Let $d' : A \times A \rightarrow \mathbb{R}_+$ defined by $d'(x, y) = d_{\mathcal{I}}(\phi(x), \phi(y))$ for all $x, y \in A$. Then d' is induced by some norm. Therefore $d'_{(A \cap B)^2} = d_{\mathcal{I}|(A \cap B)^2}$ and as $A \cap B$ contains an open set, $d' = d_{\mathcal{I}}$ and thus $\phi : (A, d_{\mathcal{I}}) \rightarrow (B, d_{\mathcal{I}})$ is an isometry.

Let M be a wall of \mathbb{A} , D_1 and D_2 be the half-apartments defined by M and $s \in W^a$ be the reflection with respect to M . Let B be an apartment of \mathcal{I} such that $\mathbb{A} \cap B = D$, which exists by Lemma 3.10. Let D_3 be the half-apartment of A_1 opposite to D_1 . Then $D_3 \cap D_2 \subset D_3 \cap A \subset M$ and thus $D_2 \cap D_3 = M$. By Proposition 2.9 2) of [Rou11], $D_3 \cup D_2$ is an apartment A_2 of \mathcal{I} . Let $\phi_1 : \mathbb{A} \xrightarrow{A_2 \cap A_1} A_1$, $\phi_2 : \mathbb{A} \xrightarrow{A_2 \cap A_1} A_2$ and $\phi_3 : A_1 \xrightarrow{A_2 \cap A_1} A_2$. Then by Lemma 3.4, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{s} & \mathbb{A} \\ \downarrow \phi_2 & & \downarrow \phi_1 \\ A_2 & \xrightarrow{\phi_3} & A_1 \end{array}.$$

Therefore s is an isometry and thus W^a is a group of isometries for $d_{\mathcal{I}|A^2}$. By Proposition 2.3 1, W^v is finite and by [Rou11] 2.2 6), \mathcal{I} is a building. \square

Lemma 3.12. *Let \mathfrak{q} be a sector-germ of \mathcal{I} and d be a distance on \mathcal{I} inducing the affine topology on each apartment and such that there exists a continuous retraction of \mathcal{I} centered at \mathfrak{q} . Then each apartment containing \mathfrak{q} is closed.*

Proof. Let A be an apartment containing \mathfrak{q} and $B = \rho(\mathcal{I})$. Let $\phi : B \xrightarrow{\mathfrak{q}} A$ and $\rho_A : \mathcal{I} \xrightarrow{\mathfrak{q}} A$. Then $\rho_A = \phi \circ \rho$ is continuous because ϕ is an affine map. Let $(x_n) \in A^{\mathbb{N}}$ be a converging sequence and $x = \lim x_n$. Then $x_n = \rho_A(x_n) \rightarrow \rho_A(x)$ and thus $x = \rho(x) \in A$. \square

Proposition 3.13. *Suppose \mathcal{I} is not a building. Let d be a distance on \mathcal{I} inducing the affine topology on each apartment. Then the interior of each apartment of \mathcal{I} is empty.*

Proof. Let U be a nonempty open set of \mathcal{I} . Let A be an apartment of \mathcal{I} such that $A \cap U \neq \emptyset$. By Proposition 2.3 2, there exists a true wall M of A such that $D \cap U \neq \emptyset$. Let D be a half-apartment D delimited by M . Let B be an apartment such that $A \cap B = D$, which exists by Lemma 3.10. Then $B \cap U$ is an open set of B containing $M \cap U$ and thus $E \cap U \neq \emptyset$, where E is the half-apartment of B opposite to D . Therefore $U \setminus A \neq \emptyset$ and we get the proposition. \square

One sets $\mathcal{I}_0 = G.0$ where $0 \in \mathbb{A}$. This is the set of *vertices of type 0*. Let $-\infty = \text{germ}_{\infty}(-C_f^v)$. One sets $\rho_{-\infty} : \mathcal{I} \xrightarrow{-\infty} \mathbb{A}$.

Recall the definition of the lattice Y in Subsection 2.1).

Lemma 3.14. *One has $\mathcal{I}_0 \cap \mathbb{A} = Y$.*

Proof. Let $x \in \mathcal{I}_0 \cap \mathbb{A}$. One has $x = g.0$ with $g \in G$. By (MA2), there exists $\phi : g.\mathbb{A} \rightarrow \mathbb{A}$ fixing x . Then $x = \phi(g.x)$ and $\phi \circ g|_{\mathbb{A}} : \mathbb{A} \rightarrow \mathbb{A}$ is an automorphism of apartment and hence $x \in Y$ (by the end of Subsection 2.5). \square

Lemma 3.15. *The set \mathcal{I}_0 is countable.*

Proof. Let $i \in \{-\infty, +\infty\}$. By definition of ρ_i , $\rho_i(x) \in \mathcal{I}_0$ for all $x \in \mathcal{I}_0$ and thus $\rho_i(x) \in Y$ for all $x \in \mathcal{I}_0$. Therefore $\mathcal{I}_0 = \bigcup_{(\lambda, \mu) \in Y^2} \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})$. By Theorem 5.6 of [Héb16], $\rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})$ is finite for all $(\lambda, \mu) \in Y^2$, which completes the proof. \square

Let \mathfrak{q} be a sector-germ of \mathcal{I} . For all $z \in \mathcal{I}_0$, one chooses an apartment $A(z)$ containing z and \mathfrak{q} . Let $x \in \mathcal{I}$ and A be an apartment containing x and \mathfrak{q} . There exists $z \in \mathcal{I}_0 \cap A$ such that $x \in z + \mathfrak{q}$ and thus $x \in A(z)$. Therefore $\mathcal{I} = \bigcup_{z \in \mathcal{I}_0} A(z)$.

Proposition 3.16. *Let d be a distance on \mathcal{I} . Suppose that there exists a sector-germ \mathfrak{q} such that each apartment containing \mathfrak{q} is closed and with empty interior. Then (\mathcal{I}, d) is not complete and the interior of each compact of \mathcal{I} is empty.*

Proof. One has $\mathcal{I} = \bigcup_{z \in \mathcal{I}_0} A(z)$, with \mathcal{I}_0 countable by Lemma 3.15. Thus by Baire's Theorem, (\mathcal{I}, d) is not complete.

Let K be a compact of \mathcal{I} and $U \subset K$ be open. Then \overline{U} is compact and thus complete. One has $\overline{U} = \bigcup_{z \in \mathcal{I}_0} \overline{U} \cap A(z)$ and thus \overline{U} has empty interior. Thus K has empty interior. \square

4 Construction of signed distances

4.1 Definition of positive and negative distances

In this section we construct distances on \mathcal{I} . To each sector-germ \mathfrak{q} and to each norm on an apartment containing \mathfrak{q} , we associate a distance on \mathcal{I} . Let us be more precise.

Let A be an apartment of \mathcal{I} . Let d be a distance on A . One says that d is a norm if there exists an isomorphism $\phi : \mathbb{A} \rightarrow A$ and a norm $|\cdot|$ on \mathbb{A} such that $d(x, y) = |\phi^{-1}(x) - \phi^{-1}(y)|$ for all $(x, y) \in A^2$. A distance d on A is a norm if and only if for all isomorphism $\phi : \mathbb{A} \rightarrow A$, there exists a norm $|\cdot|$ on \mathbb{A} such that $d(x, y) = |\phi^{-1}(x) - \phi^{-1}(y)|$ for all $x, y \in A$. Let $\mathcal{N}(A)$ be the set of norms on A . Let \mathfrak{q} be a sector-germ of \mathcal{I} and $\mathcal{A}(\mathfrak{q})$ be the set of apartments containing \mathfrak{q} . Let $A \in \mathcal{A}(\mathfrak{q})$ and $d \in \mathcal{N}(A)$. Then one sets $\theta_{\mathfrak{q}}(d) = (d_B)_{B \in \mathcal{A}(\mathfrak{q})}$, where for all $B \in \mathcal{A}(\mathfrak{q})$ and all $x, y \in B$, $d_B(x, y) = d(\phi^{-1}(x), \phi^{-1}(y))$, where $\phi : B \xrightarrow{A \cap B} A$. Let $\Theta(\mathfrak{q}) = \{\theta_{\mathfrak{q}}(d) \mid d \in \bigcup_{A \in \mathcal{A}(\mathfrak{q})} \mathcal{N}(A)\}$. In this section, we fix \mathfrak{q} and we associate to each $\theta \in \Theta(\mathfrak{q})$ a distance d_{θ} on \mathcal{I} .

4.2 Translation in a direction

In this subsection we define for all sector Q of direction \mathfrak{q} a map $+: \mathcal{I} \times \overline{Q}$ such that for all $x \in \mathcal{I}$ and $q \in \overline{Q}$, $x + q$ is the "translate of x by q ".

Let $A \in \mathcal{A}(\mathfrak{q})$. One chooses an isomorphism $\phi : \mathbb{A} \rightarrow A$. We consider A as a vectorial space over \mathbb{R} via this isomorphism. Let $Q = 0 + \mathfrak{q} \subset A$. Let $\rho : \mathcal{I} \xrightarrow{A} A$.

Lemma 4.1. *Let $x \in \mathcal{I}$. Then $\rho_x : x + \overline{Q} \rightarrow \rho(x) + \overline{Q}$ is well defined and is a bijection.*

Proof. Let B be an apartment containing $x + \overline{Q}$ and $\phi : B \xrightarrow{A} A$. Then $\phi(x + \overline{Q}) = \phi(x) + \overline{Q} = \rho(x) + \overline{Q}$. Let $u, u' \in x + \overline{Q}$ be such that $\rho(u) = \rho(u')$. Then $u, u' \in B$, thus $\phi(u) = \phi(u') = \rho(u) = \rho(u')$ and thus $u = u'$. \square

Let $x \in \mathcal{I}$ and $u \in \overline{Q}$. One sets $x + u = \rho_x^{-1}(\rho(x) + u)$. If B is an apartment containing x and \mathfrak{q} , then for all $u \in \overline{Q}$, $x + u \in B$ (because $B \supset x + \mathfrak{q}$).

Lemma 4.2. *Let $x \in \mathcal{I}$. Then for all $u, v \in \overline{Q}$, $(x+u)+v = x+(u+v)$ and $x+u+v = x+v+u$.*

Proof. Let B be an apartment containing \mathfrak{q} and x . Let $\phi : A \xrightarrow{A \cap B} B$. This isomorphism enables to consider B as an affine space under the action of A . For $u \in \overline{Q}$, one denotes by $\tau_u : B \rightarrow B$ the translation of vector u . Then for all $u \in \overline{Q}$, $\tau_u(x) = x + u$, which proves the lemma. \square

4.3 Definition of a distance

Let $\theta = (d_B)_{B \in \mathcal{A}(\mathfrak{q})} \in \Theta(\mathfrak{q})$. For all $u \in A$, one sets $|u| = d_A(u, 0)$. For $x, y \in \mathcal{I}$, one sets $T(x, y) = \{(u, v) \in \overline{Q}^2 \mid x + u = y + v\}$. One defines $d_{\theta}(x, y) = \inf_{(u, v) \in T(x, y)} |u| + |v|$. Until the end of this section, we will write d instead of d_{θ} .

Let $\mathbb{A}_{in} = \bigcap_{i \in I} \ker \alpha_i \subset \mathbb{A}$ (where the α_i are the simple roots of \mathbb{A}). Let $A_{in} = \phi(\mathbb{A}_{in})$.

Lemma 4.3. *Let $a, b \in \mathcal{I}$. Then $T(a, b)$ is not empty.*

Proof. Let $(e_i)_{i \in J}$ be a basis of A such that for some $J' \subset J$, $(e_i)_{i \in J \setminus J'}$ is a basis of A_{in} and such that $Q = A_{in} \oplus \bigoplus_{i \in J'} \mathbb{R}_+^* e_i$. Let (e_i^*) be the dual basis of (e_i) . Let $B \in \mathcal{A}(\mathfrak{q})$ and

$\phi : A \xrightarrow{q} B$. For $i \in J$, one defines $e_{i,B}^* : B \rightarrow \mathbb{R}$ by $e_{i,B}^*(x) = e_i^*(\phi^{-1}(x))$ for all $x \in B$. Then for $i \in J$, $x \in B$ and $q \in Q$, $e_{i,B}^*(x+q) = e_{i,B}^*(x) + e_i^*(q)$.

For $x \in \mathcal{I}$ one chooses $B_x \in \mathcal{A}(\mathfrak{q})$ containing x . Let $Q_x \subset Q$ be a sector of direction \mathfrak{q} included in $A \cap B_x$ and $M_x \in \mathbb{R}$ such that for all $y \in A$, $\min_{i \in J'} e_i^*(y) \geq M_x$ implies $y \in Q_x \subset A \cap B_x$. One chooses $q_x \in Q$ such that $\min_{i \in J'} (e_{i,B_x}^*(x) + e_i^*(q_x)) \geq M_x$. Then $x + q_x \in A$ for all $x \in \mathcal{I}$. Therefore $a + q_a$ and $b + q_b$ are in A . Thus there exists $r, s \in Q$ such that $a + q_a + r = b + q_b + s$ and $(q_a + r, q_b + s) \in T(a, b)$. \square

Proposition 4.4. *The function $d : \mathcal{I}^2 \rightarrow \mathbb{R}_+$ is a distance.*

Proof. The function d is clearly symmetric. Let us show the triangular inequality. Let $x, y, z \in \mathcal{I}$. Let $\epsilon > 0$ and $(u, v) \in T(x, y)$, $(\mu, \nu) \in T(y, z)$ be such that $|u| + |v| \leq d(x, y) + \epsilon$ and $|\mu| + |\nu| \leq d(y, z) + \epsilon$. One has $x + u = y + v$ and $y + \mu = z + \nu$. Thus $x + u + \mu = y + \mu + v = z + \nu + v$ (by Lemma 4.2) and hence $(u + \mu, \nu + v) \in T(x, z)$. Consequently, $d(x, z) \leq |u + \mu| + |\nu + v| \leq |u| + |v| + |\mu| + |\nu| \leq d(x, y) + d(y, z) + 2\epsilon$, which proves the triangular inequality. Let $x, y \in \mathcal{I}$ be such that $d(x, y) = 0$. Let $(u_n, v_n) \in T(x, y)^{\mathbb{N}}$ be such that $u \rightarrow 0$ and $v \rightarrow 0$. Let $n \in \mathbb{N}$. One has $y + \mathfrak{q} \supset y + v_n + \mathfrak{q} = x + u_n + \mathfrak{q}$ and thus $y + \mathfrak{q} \supset \bigcup_{n \in \mathbb{N}} x + u_n + \mathfrak{q} = x + \mathfrak{q}$. By symmetry, $x + \mathfrak{q} \supset y + \mathfrak{q}$ and hence $x + \mathfrak{q} = y + \mathfrak{q}$. Let B be an apartment containing x and \mathfrak{q} . By (MA2), $B \supset cl(x + \mathfrak{q}) = cl(y + \mathfrak{q}) \ni y$. Therefore, $x = y$. \square

One equips $\mathcal{I} \times \overline{Q}$ with a distance d defined by $d((x, q), (x', q')) = d(x, x') + |q - q'|$.

Lemma 4.5. *The map $\begin{matrix} \mathcal{I} \times \overline{Q} \rightarrow \mathcal{I} \\ (x, \nu) \mapsto x + \nu \end{matrix}$ is Lipschitz continuous.*

Proof. By the fact that each norm are equivalent on an affine space of finite dimension, one can choose a particular $\theta \in \Theta(\mathfrak{q})$. Let $(e_i)_{i \in J}$ be a basis of A such that for some $J' \subset J$, $(e_i)_{i \in J \setminus J'}$ is a basis of A_{in} and $Q = A_{in} \oplus \bigoplus_{i \in J \setminus J'} \mathbb{R}_+^* e_i$ and (e_i^*) be the dual basis to (e_i) . For $x \in A$, one sets $|x| = \sum_{i \in I} |e_i^*(x)|$ and one supposes that θ is associated to $|\cdot|$. Let us show that $+$ is 1-Lipschitz continuous.

Let $x, x' \in \mathcal{I}$ and $\epsilon > 0$. Let $(u, u') \in T(x, x')$ such that $d(x, x') \leq |u| + |u'| + \epsilon$. Let $\nu, \nu' \in \overline{Q}$. One sets $\lambda_i = e_i^*(\nu - \nu')$ for all $i \in J$. Let $U = \{i \in J \mid \lambda_i > 0\}$ and $V = \{i \in J \mid \lambda_i < 0\}$. Let $\mu' = \sum_{i \in U} \lambda_i e_i$ and $\mu = -\sum_{i \in V} \lambda_i e_i$. One has $\nu + \mu = \nu' + \mu'$ and $|\mu| + |\mu'| = d(\nu, \nu')$. One has $x + \nu + u + \mu = x' + \nu' + u' + \mu'$ and thus $d(x + \nu, x' + \nu') \leq |u| + |u'| + |\mu| + |\mu'| \leq d(x, x') + d(\nu, \nu') + \epsilon$, which enables to conclude. \square

Remark 4.6. A consequence of Lemma 4.5 is the fact that for all $x, y \in \overline{Q}$, $d(x, y) = \min_{u, v \in T(x, y)} |u| + |v|$.

Proposition 4.7. *For all $x, y \in \mathcal{I}$, there exists a geodesic from x to y .*

Proof. Let $x, y \in \mathcal{I}$ and $(u, v) \in T(x, y)$ such that $d(x, y) = |u| + |v|$. One defines $\tau : [0, 1] \rightarrow \mathcal{I}$ by $\tau(t) = x + 2tu$ if $t \in [0, \frac{1}{2}]$ and $\tau(t) = y + 2(1-t)v$ if $t \in [\frac{1}{2}, 1]$, and τ is a geodesic from x to y . \square

Remark 4.8. If $\dim \mathbb{A} \geq 2$, for any choice of θ , there exists two points in \mathcal{I} such that there exist infinitely many geodesics between them. For example, if we choose the norm on A as in the proof of Lemma 4.5, we have that $d_{|\mathbb{A}^2|}$ is the distance induced by $|\cdot|$. If $x = \sum_{i \in I} x_i e_i \in A$ and if for all $i \in I$, $f_i : [0, 1] \rightarrow \mathbb{R}$ is a continuous monotonic function such that $f_i(0) = 0$ and $f_i(1) = x_i$, then $f = (f_i)_{i \in I}$ is a geodesic from 0 to x .

Lemma 4.9. *Let B and C be two apartments containing \mathbf{q} . Then:*

(i) *the retraction $\rho_B : (\mathcal{I}, d) \xrightarrow{q} (B, d|_{B^2})$ is 1-Lipschitz continuous.*

(ii) *the isomorphism $\varphi : (B, d|_{B^2}) \xrightarrow{q} (C, d|_{C^2})$ is an isometry.*

Proof. Let $\rho : \mathcal{I} \xrightarrow{q} A$. Let $x, y \in \mathcal{I}$ and $(u, v) \in T(x, y)$. One has $x + u = y + v$, thus $\rho(x + u) = \rho(x) + u = \rho(y + v) = \rho(y) + v$ and hence $(u, v) \in T(\rho(x), \rho(y))$, which proves that ρ is 1-Lipschitz continuous.

First suppose that $C = A$. Let $x, y \in B$ and $(u, v) \in T(\rho(x), \rho(y))$. Then $x + u = \rho_x^{-1}(\rho(x) + u) = \varphi^{-1}(\rho(x) + u) = \varphi^{-1}(\rho(y) + v) = y + v$. Hence $T(x, y) \supset T(\rho(x), \rho(y))$ and thus $T(x, y) = T(\rho(x), \rho(y))$. Therefore, φ is an isometry. Suppose now $C \neq A$. Let $\varphi_1 : B \xrightarrow{q} A$, $\varphi_2 : A \xrightarrow{q} C$ and $\varphi : B \xrightarrow{q} C$. Then $\varphi = \varphi_2 \circ \varphi_1$ is an isometry. One has $\rho_B = \varphi_1^{-1} \circ \rho$ and thus ρ_B is 1-Lipschitz continuous. \square

4.4 Independence of the choices of apartments and isomorphisms

Let us show that the distance we defined only depends on $\theta \in \Theta(\mathbf{q})$. For this we have to show that d is independent of the choice of $A \in \mathcal{A}(\mathbf{q})$ and of the isomorphism $\phi : \mathbb{A} \rightarrow A$. Let $A' \in \mathcal{A}(\mathbf{q})$, $f : A \xrightarrow{q} A'$ and $\phi' = f \circ \phi$. One considers A' as a vectorial space over \mathbb{R} by saying that f is an isomorphism of vectorial space. Objects or operations in A' are denoted with a $'$.

Lemma 4.10. *Let $x \in \mathcal{I}$ and $u \in \overline{Q}$. Then $x + u = x + ' f(u)$.*

Proof. One has $\rho'(x + u) = f \circ \rho(x + u) = f(\rho(x) + u) = f(\rho(x)) + ' f(u) = \rho'(x) + ' f(u)$. Let B be an apartment containing x and \mathbf{q} . Then $B \ni x + u, x + ' f(u)$ and thus $x + u = x + ' f(u)$. \square

Lemma 4.11. *One has $d = d'$.*

Proof. Let $x, y \in \mathcal{I}$. By Lemma 4.10, $T_{A'}(x, y) = f(T_A(x, y))$. Let $u \in A$. Then $|f(u)'| = d_{A'}(0', f(u)) = d_{A'}(f(0), f(u)) = d_A(0, u) = |u|$ by definition of $\Theta(\mathbf{q})$. Therefore, $d = d'$. \square

Let now $\psi : \mathbb{A} \rightarrow A$ be an other isomorphism of apartments. This defines an other structure of vectorial space on A . We put a subscript ψ or ϕ to make the difference between these structures.

Lemma 4.12. *One has $d_\phi = d_\psi$.*

Proof. One has $\psi = \phi \circ w$ with $w \in W$. One writes $w = \tau \circ \vec{w}$, where \vec{w} is the vectorial part of w and τ is a translation of \mathbb{A} . Let us show that if $\tilde{\tau} = \phi \circ \tau^{-1} \circ \phi^{-1}$, $x +_\psi u = x +_\phi \tilde{\tau}(u)$ for all $(x, u) \in \mathcal{I} \times \overline{Q_\psi}$. Let $x, u \in A$. One has $x +_\psi u = \psi(\psi^{-1}(x) + \psi^{-1}(u)) = \phi(\phi^{-1}(x) + \vec{w}(w^{-1} \circ \phi^{-1}(u))) = \phi(\phi^{-1}(x) + \phi^{-1}(\tilde{\tau}(u))) = x +_\phi \tilde{\tau}(u)$. Therefore, for all $(x, u) \in \mathcal{I} \times \overline{Q_\psi}$, $x +_\psi u = \rho_x^{-1}(x +_\psi u) = \rho_x^{-1}(x +_\phi \tilde{\tau}(u)) = x +_\phi \tilde{\tau}(u)$. Let us show that $|u|_\psi = |\tilde{\tau}(u)|_\phi$ for all $u \in \overline{Q_\psi}$. Let $x \in A$. Then $x +_\psi 0_\psi = x +_\phi \tilde{\tau}(0_\psi) = x +_\phi 0_\phi$ and thus $0_\phi = \tilde{\tau}(0_\psi)$. Let $u \in A$. Then $|u|_\psi = d(u, 0_\psi) = d(\tilde{\tau}(u), \tilde{\tau}(0_\psi)) = d(\tilde{\tau}(u), 0_\phi) = |\tilde{\tau}(u)|_\phi$ because $\tilde{\tau}$ is a translation of A . Let $x, y \in \mathcal{I}$. Then $T_\psi(x, y) = \tilde{\tau}(T_\phi(x, y))$ and thus $d_\psi(x, y) = d_\phi(x, y)$. \square

Thus we have constructed a distance d_θ for all $\theta \in \Theta(\mathbf{q})$. When \mathcal{I} is a tree, we obtain the usual distance.

5 Comparison of distances of the same sign

The aim of this section is to show that if \mathfrak{q} and \mathfrak{q}' are sector-germs of \mathcal{I} of the same sign and if $\theta \in \Theta(\mathfrak{q})$, $\theta' \in \Theta(\mathfrak{q}')$ then d_θ and $d_{\theta'}$ are equivalent, which means that there exists $k, l \in \mathbb{R}_+^*$ such that $kd_\theta \leq d_{\theta'} \leq ld_\theta$. To prove this we make an induction on the distance between \mathfrak{q} and \mathfrak{q}' . In the next subsection, we treat the case where \mathfrak{q} and \mathfrak{q}' are adjacent. We use results of Subsection 3.1.

Lemma 5.1. *Let \mathfrak{q} be a sector germ of \mathcal{I} . Then for all $\theta \in \Theta(\mathfrak{q})$ and all $A \in \mathcal{A}(\mathfrak{q})$, $d_{\theta|A^2} \in \mathcal{N}(A)$*

Proof. Let $A \in \mathcal{A}(\mathfrak{q})$ and $\theta \in \Theta(\mathfrak{q})$. One equips A with a structure of vectorial space. Let $N : A \rightarrow \mathbb{R}_+$ defined by $N(x) = d_\theta(x, 0)$ for all $x \in A$. Let $x, y \in A$. Then $T(x, y) = T(0, y - x)$ and thus $d_\theta(x, y) = N(y - x)$. Let $\lambda \in \mathbb{R}^*$ and $x \in A$. Then $T(0, \lambda x) = |\lambda|T(0, x)$ and thus $N(\lambda x) = |\lambda|N(x)$. \square

5.1 Comparison of distance for adjacent sector-germs

Let A be an apartment of \mathcal{I} and $\mathfrak{q}, \mathfrak{q}'$ be two adjacent sector-germs of A . Let d be a norm on A , $\theta = \theta_{\mathfrak{q}}(d)$ and $\theta' = \theta_{\mathfrak{q}'}(d)$. Let $d_{\mathfrak{q}} = d_\theta$ and $d_{\mathfrak{q}'} = d_{\theta'}$. We fix a vectorial structure on A . One sets $|x| = d(0, x)$ for all $x \in A$. Let $Q = 0 + \mathfrak{q}$ and $Q' = 0 + \mathfrak{q}'$. For all $x, y \in \mathcal{I}$ and $R \in \{Q, Q'\}$, one sets $T_R(x, y) = \{u, v \in \overline{R} \mid x + u = y + v\}$. The aim of this subsection is to show that there exists $k \in \mathbb{R}$ such that $d_{\mathfrak{q}} \leq kd_{\mathfrak{q}'}$. Let $\rho_{\mathfrak{q}} : \mathcal{I} \xrightarrow{\mathfrak{q}} A$ and $\rho_{\mathfrak{q}'} : \mathcal{I} \xrightarrow{\mathfrak{q}'} A$.

Lemma 5.2. *There exists $l \in \mathbb{R}_+^*$ such that for all $B \in \mathcal{A}(\mathfrak{q}) \cap \mathcal{A}(\mathfrak{q}')$ we have: for all $x, y \in B$, $d_{\mathfrak{q}'}(x, y) \leq ld_{\mathfrak{q}}(x, y)$.*

Proof. Let $x, y \in B$. By Lemma 4.9, $d_{\mathfrak{q}}(x, y) = d_{\mathfrak{q}}(\rho_{\mathfrak{q}}(x), \rho_{\mathfrak{q}}(y))$ and $d_{\mathfrak{q}'}(\rho_{\mathfrak{q}'}(x), \rho_{\mathfrak{q}'}(y)) = d_{\mathfrak{q}'}(x, y)$. As $\rho_{\mathfrak{q}'}|_B = \rho_{\mathfrak{q}}|_B$, one can suppose that $x, y \in A$. Then this is a consequence of Lemma 5.1. \square

Let B be an apartment containing \mathfrak{q} but not \mathfrak{q}' . Let \mathfrak{F}_∞ be the direction of sector-panel common to \mathfrak{q} and \mathfrak{q}' . Let $x \in B$ and N be a wall containing $x + \mathfrak{F}_\infty$. By Lemma 3.6, one can write $B = D_1 \cup D_2$, with D_1 and D_2 two opposite half-apartments having a wall H parallel to N , such that D_i and \mathfrak{q}' are included in some apartment B_i , for all $i \in \{1, 2\}$. One supposes that $D_1 \supset \mathfrak{q}$.

Let M be the wall of A containing $0 + \mathfrak{F}_\infty$, $t_0 : A \rightarrow A$ be the reflection with respect to this wall and $T \in \mathbb{R}_+$ such that $t_0 : (A, d_{\mathfrak{q}}) \rightarrow (A, d_{\mathfrak{q}})$ is T -Lipschitz continuous (such a T exists by Lemma 5.1). As t_0 is an involution, $T \geq 1$.

Lemma 5.3. *There exists a translation τ of A such that if $\tilde{t} = \tau \circ t_0$, one has for all $x \in B$:*

- if $x \in D_1$, $\rho_{\mathfrak{q}}(x) = \rho_{\mathfrak{q}'}(x)$
- if $x \in D_2$, $\rho_{\mathfrak{q}}(x) = \tilde{t} \circ \rho_{\mathfrak{q}'}(x)$

Proof. Let $\phi_i : B \xrightarrow{B \cap B_i} B_i$, for $i \in \{1, 2\}$ and $\phi : B_2 \xrightarrow{B_1 \cap B_2} B_1$. Let t be the reflection of B_1 with respect to H . By Lemma 3.4, one has the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\phi_2} & B_2 \\ \downarrow \phi_1 & & \downarrow \phi \\ B_1 & \xrightarrow{t} & B_1 \end{array}$$

Let $x \in D_1$. Then $\rho_{\mathbf{q}, B_1}(x) = x = \rho_{\mathbf{q}', B_1}(x)$. Let $\psi : B_1 \xrightarrow{B_1 \cap A} A$. Then $\rho_{\mathbf{q}}(x) = \psi(\rho_{\mathbf{q}, B_1}(x)) = \psi(\rho_{\mathbf{q}', B_1}(x)) = \rho_{\mathbf{q}'}(x)$.

Let $x \in D_2$. One has $\rho_{\mathbf{q}, B_1}(x) = \phi_1(x)$ and $\rho_{\mathbf{q}', B_1}(x) = \phi(x)$ and thus $\rho_{\mathbf{q}, B_1}(x) = t \circ \rho_{\mathbf{q}', B_1}(x)$. Let \tilde{t} making the following diagram commute:

$$\begin{array}{ccc} B_1 & \xrightarrow{t} & B_1 \\ \downarrow \psi & & \downarrow \psi \\ A & \xrightarrow{\tilde{t}} & A \end{array}$$

Then $\rho_{\mathbf{q}}(x) = \tilde{t} \circ \rho_{\mathbf{q}'}(x)$. Moreover \tilde{t} fixes $\psi(H)$, which is parallel to M . Therefore, $\tilde{t} = \tau \circ t_0$ for some translation τ of A (by Lemma 3.3). □

Lemma 5.4. *Let l be as in Lemma 5.2. Let $x, y \in B$ be such that $x, y \in D_i$ for some $i \in \{1, 2\}$. Then $d_{\mathbf{q}'}(x, y) \leq lT d_{\mathbf{q}}(x, y)$.*

Proof. By Lemma 4.9, $d_{\mathbf{q}}(x, y) = d_{\mathbf{q}}(\rho_{\mathbf{q}}(x), \rho_{\mathbf{q}}(y))$ and $d_{\mathbf{q}'}(x, y) = d_{\mathbf{q}'}(\rho_{\mathbf{q}'}(x), \rho_{\mathbf{q}'}(y))$. Lemma 5.3 completes the proof. □

Lemma 5.5. *Let (X, d) be a metric space, $f : (\mathcal{I}, d_{\mathbf{q}}) \rightarrow (X, d)$ be a map and $k \in \mathbb{R}_+$. Then f is k -Lipschitz continuous if and only if for all apartment A containing \mathbf{q} , $f|_A$ is k -Lipschitz continuous.*

Proof. One implication is clear. Suppose that for all apartment A containing \mathbf{q} , $f|_A$ is k -Lipschitz continuous. Let $x, y \in \mathcal{I}$. Let $(u, v) \in T(x, y)$ such that $|u| + |v| = d(x, y)$. One has $d(f(x), f(y)) \leq d(f(x), f(x+u)) + d(f(y+v), f(y)) \leq k(|u| + |v|) \leq kd(x, y)$. □

Lemma 5.6. *One has $d_{\mathbf{q}'} \leq lT d_{\mathbf{q}}$.*

Proof. Let us prove that $\text{Id} : (\mathcal{I}, d_{\mathbf{q}}) \rightarrow (\mathcal{I}, d_{\mathbf{q}'})$ is lT -Lipschitz continuous. Let $x, y \in B$. We already know that if $x, y \in D_i$ for some $i \in \{1, 2\}$, $d_{\mathbf{q}'}(x, y) \leq lT d_{\mathbf{q}}(x, y)$. Suppose now $x \in D_1$ and $y \in D_2$. Let $u \in [x, y] \cap H$. By Lemma 5.1, one has:

$$d_{\mathbf{q}'}(x, y) \leq d_{\mathbf{q}'}(x, u) + d_{\mathbf{q}'}(u, y) \leq lT(d_{\mathbf{q}}(x, u) + d_{\mathbf{q}}(u, y)) = lT d_{\mathbf{q}}(x, y).$$

As B is an arbitrary apartment containing \mathbf{q} and not \mathbf{q}' , Lemma 5.2 and Lemma 5.5 completes the proof. □

5.2 Comparison of distances for sector-germs of the same signs

In this subsection, we show that if \mathbf{q} and \mathbf{q}' are sector-germs of the same signs, d_{θ} and $d_{\theta'}$ are equivalent. We then deduce corollaries about the induced topologies on each apartment and on retractions centered at a sector-germ.

Let Θ be the disjoint union of the $\Theta(\mathfrak{s})$ for \mathfrak{s} a sector-germ of \mathcal{I} . Let $\theta \in \Theta$, \mathbf{q} such that $\theta \in \Theta(\mathbf{q})$ and $\epsilon \in \{-, +\}$. One says that θ is of sign ϵ if \mathbf{q} is of sign ϵ . For $\epsilon \in \{-, +\}$, one denotes by Θ_{ϵ} the set of elements of Θ of sign ϵ .

Theorem 5.7. *Let $\mathbf{q}_1, \mathbf{q}_2$ be two sector germs of \mathcal{I} of the same sign. For $i \in \{1, 2\}$, let $\theta_i \in \Theta(\mathbf{q}_i)$. Then there exists $k \in \mathbb{R}_+^*$ such that $d_{\theta_1} \leq k d_{\theta_2}$. In particular for all $\theta \in \Theta$, the topology of $(\mathcal{I}, d_{\theta})$ only depends on the sign of θ .*

Proof. Let A be an apartment containing \mathbf{q}_1 and \mathbf{q}_2 , and $d \in \mathcal{N}(A)$. Let $\mathbf{s}_0 = \mathbf{q}_1, \dots, \mathbf{s}_n = \mathbf{q}_2$ be a gallery joining \mathbf{q}_1 and \mathbf{q}_2 . For all $i \in \llbracket 0, n \rrbracket$, one sets $u_i = \theta_{\mathbf{s}_i}(d)$. By an induction using Lemma 5.6, there exists $a \in \mathbb{R}_+^*$ such that $d_{u_0} \leq ad_{u_n}$. As every norms are equivalent on A , there exists $b, c \in \mathbb{R}_+^*$ such that $d_{\theta_1} \leq bd_{u_0}$ and $d_{u_n} \leq cd_{\theta_2}$, which concludes the proof of the theorem. \square

We thus obtain (at most) two topologies on \mathcal{I} : the topology \mathcal{T}_+ obtained by taking a positive $\theta \in \Theta$ and the topology \mathcal{T}_- obtained by taking a negative $\theta \in \Theta$. We will see that when \mathcal{I} is not a building, these topologies are different.

Corollary 5.8. *Let A be an apartment of \mathcal{I} and $\theta \in \Theta$. Then the topology of A induced by the topology of (\mathcal{I}, d_θ) is the affine topology on A .*

Proof. By Theorem 5.7, one can suppose $\theta \in \Theta(\mathbf{q})$ where \mathbf{q} is a sector germ of A of the same sign as θ . Then Lemma 5.1 concludes the proof. \square

Corollary 5.9. *Let \mathbf{q} be a sector-germ. Let ρ be a retraction with center \mathbf{q} , $A = \rho(\mathcal{I})$ and $d \in \mathcal{N}(A)$. Then:*

- (i) *for each $\theta \in \Theta$ of the sign of \mathbf{q} , $\rho : (\mathcal{I}, d_\theta) \rightarrow (A, d)$ is Lipschitz continuous*
- (ii) *if B is an apartment and $d' \in \mathcal{N}(B)$, $\rho|_B : (B, d') \rightarrow (A, d)$ is Lipschitz continuous.*

Proof. Let $\theta \in \Theta(\mathbf{q})$. Then by Lemma 4.9, $\rho : (\mathcal{I}, d_\theta) \rightarrow (A, d_\theta)$ is Lipschitz continuous and one concludes the proof of (i) with Theorem 5.7 and Lemma 5.1.

Let \mathbf{q}' be a sector-germ of B of the same sign as \mathbf{q} and $\theta' \in \Theta(\mathbf{q}')$. Then $\rho : (\mathcal{I}, d_{\theta'}) \rightarrow (A, d)$ is Lipschitz continuous by (i). Thus $\rho|_B : (B, d_{\theta'}) \rightarrow (A, d)$ is Lipschitz continuous and one concludes with Lemma 5.1. \square

Corollary 5.10. *Let A, B be two apartments of \mathcal{I} . Then $A \cap B$ is a closed subset of A (seen as an affine space).*

Proof. By Lemma 3.12, A and B are closed for \mathcal{T}_+ (or \mathcal{T}_-). Th $A \cap B$ is closed for \mathcal{T}_+ , therefore it is closed for the topology induced by \mathcal{T}_+ on A , and Corollary 5.8 completes the proof. \square

Remark 5.11. Suppose that \mathcal{I} is not a building. Then by Subsection 3.3, for all $\theta \in \Theta$, (\mathcal{I}, d_θ) is not complete.

Let \mathbf{q} be a sector-germ of \mathcal{I} and (Q_n) be an increasing sequence of sectors with germ \mathbf{q} . One says that (Q_n) is converging if there exists a retraction onto an apartment $\rho : \mathcal{I} \xrightarrow{q} \rho(\mathcal{I})$ such that $(\rho(x_n))$ converges, where x_n is the base point of Q_n for all $n \in \mathbb{N}$ and we call *limit* of (Q_n) the set $\bigcup_{n \in \mathbb{N}} Q_n$. One can show that the fact that \mathcal{I} is not complete implies that there exists a converging sequence of direction \mathbf{q} whose limit is not a sector of \mathcal{I} , which is impossible in a building. To prove this one can associate to each Cauchy sequence (x_n) a sequence (z_n) such that $d(z_n, x_n) \rightarrow 0$ and such that $z_n + \mathbf{q} \subset z_{n+1} + \mathbf{q}$ for all $n \in \mathbb{N}$. Then we show that (z_n) converges in (\mathcal{I}, d_θ) if and only if the limit of $(z_n + \mathbf{q})$ is a sector of \mathcal{I} .

5.3 Study of the action of G

In this subsection, we show that each $g \in G$, $g : \mathcal{I} \rightarrow \mathcal{I}$ is Lipschitz continuous for the distances we constructed. We begin by treating the case where g stabilizes a sector-germ. After this, we treat the case where g stabilizes an apartment and then we conclude.

Lemma 5.12. *Let $g \in G$ and suppose that g stabilizes some sector-germ \mathfrak{q} . Let $\theta \in \Theta(\mathfrak{q})$. Then $g : (\mathcal{I}, d_\theta) \rightarrow (\mathcal{I}, d_\theta)$ is an isometry.*

Proof. Let us prove that g is 1-Lipschitz continuous. Let A be an apartment containing \mathfrak{q} and $\phi : A \xrightarrow{g} g.A$. Let τ making the following diagram commute:

$$\begin{array}{ccc} (A, d_\theta) & \xrightarrow{g} & (g.A, d_\theta) \\ & \searrow \phi & \downarrow \tau \\ & & (g.A, d_\theta) \end{array}$$

Then τ is an automorphism of $g.A$ stabilizing \mathfrak{q} and thus τ is a translation. As $g.A \supset \mathfrak{q}$, τ is an isometry of $(g.A, d_\theta)$. By Lemma 4.9, ϕ is an isometry. Thus $g|_A^{g.A}$ is an isometry and it is in particular 1-Lipschitz continuous. By Lemma 5.5, g is 1-Lipschitz continuous. As g^{-1} stabilizes \mathfrak{q} , it is also 1-Lipschitz continuous, which shows the lemma. \square

Lemma 5.13. *Let A be an apartment of \mathcal{I} . We fix a structure of vectorial space on A . Let Q be a sector of \mathcal{I} with base point 0. Let $g \in G$ stabilizing A . Let $w = g|_A^A$ and \vec{w} be the linear part of w . Then for all $x \in \mathcal{I}$ and $u \in Q$, $g.(x + u) = g.x + \vec{w}(u)$.*

Proof. Let $\rho_{\mathfrak{q}} : \mathcal{I} \xrightarrow{g} A$ and $\rho_{g.\mathfrak{q}} : \mathcal{I} \xrightarrow{g} g.A$. Let us show that $\rho_{g.\mathfrak{q}} = g.\rho_{\mathfrak{q}}.g^{-1}$. Let $x \in \mathcal{I}$ and B be an apartment containing x and \mathfrak{q} . Let $\phi : B \xrightarrow{B \cap A} A$. One has $\rho_{\mathfrak{q}}(x) = \phi(x)$. As $g.\phi.g^{-1} : g.B \xrightarrow{g} g.A$, $\rho_{g.\mathfrak{q}}(g.x) = g.\phi(g^{-1}.g.x) = g.\rho_{\mathfrak{q}}(x)$. Therefore $\rho_{g.\mathfrak{q}} = g.\rho_{\mathfrak{q}}.g^{-1}$. Let $x \in \mathcal{I}$ and $u \in Q$. One has

$$\rho_{g.\mathfrak{q}}(g(x + u)) = g.\rho_{\mathfrak{q}}(x + u) = w.(\rho_{\mathfrak{q}}(x) + u) = w(\rho_{\mathfrak{q}}(x)) + \vec{w}(u).$$

We also have

$$\rho_{g.\mathfrak{q}}(g.x + \vec{w}(u)) = \rho_{g.\mathfrak{q}}(g.x) + \vec{w}(u) = g.\rho_{\mathfrak{q}}(x) + \vec{w}(u) = \rho_{g.\mathfrak{q}}(g(x + u)).$$

Then $B \ni x + u$ and $g.B \ni g.(x + u), g.x + \vec{w}(u)$. As $\rho_{g.\mathfrak{q}}|_{g.B}$ is an isomorphism, $g.(x + u) = g.x + \vec{w}(u)$. \square

Theorem 5.14. *Let $g \in G$ and $\theta \in \Theta$. Then $g : (\mathcal{I}, d_\theta) \rightarrow (\mathcal{I}, d_\theta)$ is Lipschitz continuous.*

Proof. Let \mathfrak{q} be such that $\theta \in \Theta(\mathfrak{q})$. Let A be an apartment containing \mathfrak{q} and $g.\mathfrak{q}$. Let $\phi : g.A \xrightarrow{g} A$ and $h \in G$ inducing ϕ on $g.A$. Then $g = h^{-1} \circ f$, where $f = hg$. As h^{-1} is Lipschitz continuous by Lemma 5.12 (and by Theorem 5.7), it suffices to show that f is Lipschitz continuous. One has $f(A) = A$. One fixes a vectorial structure on A . Let $x, y \in \mathcal{I}$ and \vec{w} be the linear part of $f|_A^A$.

One writes $\theta = (d_B)_{B \in \mathcal{A}(\mathfrak{q})}$. Let $\theta' = \theta_{f.\mathfrak{q}}(d_A)$. For $x \in A$, one sets $|x| = d_A(x, 0)$.

Let $k \in \mathbb{R}_+^*$ be such that $\vec{w} : (A, |\cdot|) \rightarrow (A, |\cdot|)$ is k -Lipschitz continuous. Let $x, y \in \mathcal{I}$. Let $(u, v) \in T_Q(x, y)$ such that $|u| + |v| = d_\theta(x, y)$.

By Lemma 5.13, $(\vec{w}(u), \vec{w}(v)) \in T_{\vec{w}.Q}(f(x), f(y))$. Therefore, $d_{\theta'}(f(x), f(y)) \leq |\vec{w}(u)| + |\vec{w}(v)| \leq kd_\theta(x, y)$. Therefore $f : (\mathcal{I}, d_\theta) \rightarrow (\mathcal{I}, d_{\theta'})$ is k -Lipschitz continuous and one concludes with Theorem 5.7. \square

5.4 Case of a building

In this subsection we suppose that \mathcal{I} is a building. We show that the distances we constructed and the usual distance are equivalent.

If d is a W -invariant euclidean norm on \mathbb{A} , one denotes by $d_{\mathcal{I}}$ the extension of d to \mathcal{I} in the usual manner (see [Bro89] VI.3 for example). For $x \in \mathcal{I}$, one denotes by $\text{st}(x) = \bigcup_{C \in A(x)} \overline{C}$, where $A(x)$ is the set of alcoves that contains x . Then by a Lemma in VI.3 of [Bro89], $\text{st}(x)$ contains x in its interior.

Proposition 5.15. *Let d be a W -invariant euclidean norm on \mathbb{A} and $\theta \in \Theta$. Then there exist $k, l \in \mathbb{R}_+^*$ such that $kd_{\mathcal{I}} \leq d_{\theta} \leq ld_{\mathcal{I}}$.*

Proof. Let \mathfrak{q} be such that $\theta \in \Theta(\mathfrak{q})$. By Theorem 5.7, one can suppose that $\mathfrak{q} \subset \mathbb{A}$ and that $\theta = \theta_{\mathfrak{q}}(d)$. Let $k, l \in \mathbb{R}_+^*$ such that $kd|_{A^2} \leq d_{\theta}|_{A^2} \leq ld|_{A^2}$. Let us first show that $\text{Id} : (\mathcal{I}, d_{\theta}) \rightarrow (\mathcal{I}, d_{\mathcal{I}})$ is $\frac{1}{k}$ -Lipschitz continuous. Let A be an apartment containing \mathfrak{q} . Let $x, y \in A$. Then $d_{\theta}(x, y) = d_{\theta}(\rho(x), \rho(y)) \leq \frac{1}{k}d_{\mathcal{I}}(\rho(x), \rho(y)) = \frac{1}{k}d_{\mathcal{I}}(x, y)$, where $\rho : \mathcal{I} \xrightarrow{\mathfrak{q}} \mathbb{A}$. Thus $\text{Id} : (\mathcal{I}, d_{\theta}) \rightarrow (\mathcal{I}, d_{\mathcal{I}})$ is $\frac{1}{k}$ -Lipschitz continuous by Lemma 5.5.

Let $x, y \in \mathcal{I}$. As $[x, y]$ is compact and thanks to the Lemma recalled before the proposition, there exists $n \in \mathbb{N}^*$ and $x_0 = x, x_1, \dots, x_n = y \in [x, y]$ such that $x_{i+1} \in \text{st}(x_i)$ for all $i \in \llbracket 0, n-1 \rrbracket$. In particular, for all $i \in \llbracket 0, n-1 \rrbracket$, there exists an apartment A_i containing x_i, x_{i+1} and \mathfrak{q} . One has

$$\begin{aligned} d_{\theta}(x, y) &\leq \sum_{i=0}^{n-1} d_{\theta}(x_i, x_{i+1}) = \sum_{i=0}^{n-1} d_{\theta}(\rho(x_i), \rho(x_{i+1})) \\ &\leq l \sum_{i=0}^{n-1} d_{\mathcal{I}}(\rho(x_i), \rho(x_{i+1})) = l \sum_{i=0}^{n-1} d_{\mathcal{I}}(x_i, x_{i+1}) = ld_{\mathcal{I}}(x, y), \end{aligned}$$

which proves the proposition. \square

6 Combined distances

6.1 Comparison of positive and negative topologies

In this subsection, we show that \mathcal{T}_+ and \mathcal{T}_- are different when \mathcal{I} is not a building. Let $\mathcal{I}_0 = G.0$ be the set of *vertices of type 0*. For this we prove that \mathcal{I}_0 is composed of limit points when \mathcal{I} is not a building and then we apply finiteness results of [Héb16].

Proposition 6.1. *Let $\theta \in \Theta$. Then \mathcal{I}_0 is discrete in $(\mathcal{I}, d_{\theta})$ if and only if \mathcal{I} is a building.*

Proof. Suppose that \mathcal{I} is a building. By Proposition 5.15, we can replace d_{θ} by a usual distance on \mathcal{I} . By Lemma 3.14 one has $\mathcal{I}_0 \cap \mathbb{A} = Y$, which is a lattice of \mathbb{A} . Let $\eta > 0$ such that for all $x, x' \in Y$, $d(x, x') > \eta$ implies $x = x'$. Let $x, x' \in \mathcal{I}_0$ such that $d(x, x') < \eta$. Let A be an apartment of \mathcal{I} containing x and x' and $g \in G$ such that $g.A = \mathbb{A}$. Then $d(g.x, g.x') < \eta$ and thus $x = x'$.

Suppose now that \mathcal{I} is not a building and thus that W^v is infinite. By Theorem 5.7, one can suppose that $\theta \in \Theta(\mathfrak{q})$, where \mathfrak{q} is a sector-germ of \mathbb{A} . Let $\epsilon > 0$. Let us show that there exists $x \in \mathcal{I}_0$ such that $d_{\theta}(x, 0) < 2\epsilon$ and $x \neq x'$. Let M_0 be a true wall direction of \mathbb{A} such that for all consecutive walls M_1 and M_2 of direction M_0 , $d_{\theta}(M, M') < \epsilon$ (such a direction exists by Proposition 2.3 2). Let M be a wall such that $d_{\theta}(0, M) < \epsilon$ and such that

$0 \notin D$, where D is the half apartment of \mathbb{A} delimited by M and containing \mathfrak{q} . By the proof of Proposition 3.11, there exists an apartment A such that $A \cap \mathbb{A} = D$. Let $\phi : \mathbb{A} \xrightarrow{A \cap \mathbb{A}} A$ and $x = \phi(0)$. Let $y \in M$ such that $d_\theta(0, y) < \epsilon$. Then by Lemma 4.9, $d_\theta(x, y) = d_\theta(0, y)$ and thus $d(x, 0) < 2\epsilon$. As $x \notin \mathbb{A}$, $x \neq 0$ and we get the proposition. \square

Remark 6.2. In fact, by Theorem 5.14, we have shown that when \mathcal{I} is not a building, each point of \mathcal{I}_0 is a limit point.

If B is an apartment and $(x_n) \in B^\mathbb{N}$, one says that (x_n) converges towards ∞ , if for some (for each) norm $|\cdot|$ on B , $|x_n| \rightarrow +\infty$.

Proposition 6.3. *Suppose that \mathcal{I} is not a building. Let $\epsilon \in \{-, +\}$ and $\delta = -\epsilon$. Let \mathfrak{q} (resp. \mathfrak{s}) be a sector-germ of \mathcal{I} of sign ϵ (resp. δ). Let $\theta \in \Theta(\mathfrak{q})$. We equip \mathcal{I} with d_θ . Let ρ be a retraction centered at \mathfrak{s} and $(x_n) \in \mathcal{I}_0^\mathbb{N}$ be an injective and converging sequence. Then $\rho(x_n) \rightarrow \infty$ in $\rho(\mathcal{I})$. In particular ρ is not continuous.*

Proof. By Theorem 5.7, one can suppose that there exists an apartment A containing \mathfrak{q} and \mathfrak{s} such that \mathfrak{q} and \mathfrak{s} are opposite in A . Maybe composing ρ by an isomorphism of apartments fixing \mathfrak{s} , one can suppose that $\rho(\mathcal{I}) = A$. Let $\rho_q : \mathcal{I} \xrightarrow{q} A$. Let $x = \lim x_n$ and $y = \rho_q(x)$. Then by Corollary 5.9, $\rho_q(x_n) \rightarrow y$. Let $Y_A = \mathcal{I}_0 \cap A$. Then Y_A is a lattice of A by Lemma 3.14. As $\rho_q(x_n) \in \mathcal{I}_0 \cap A$ for all $n \in \mathbb{N}$, $\rho_q(x_n) = y$ for n large enough.

We also have $\rho(x_n) \in Y_A$ for all $n \in \mathbb{N}$. By Theorem 5.6 of [Héb16], for all $a \in Y_A$, $\rho^{-1}(\{a\}) \cap \rho_q^{-1}(\{y\})$ is finite. This concludes the proof of this lemma. \square

Corollary 6.4. *If \mathcal{I} is not a building, \mathcal{T}_+ and \mathcal{T}_- are different.*

Remark 6.5. Proposition 6.3 shows that if $\theta, \theta' \in \Theta$ have opposite signs, then each open subset of (\mathcal{I}, d_θ) containing a point of \mathcal{I}_0 is not bounded for $d_{\theta'}$.

6.2 Combined distances

In this section we define and study combined distances.

Let $\Theta_c = \Theta_+ \times \Theta_-$. Let $\theta = (\theta_+, \theta_-) \in \Theta_c$. One sets $d_\theta = d_{\theta_+} + d_{\theta_-}$.

Remark 6.6. Let $\theta \in \Theta_c$. Let $f : \mathcal{I} \rightarrow \mathcal{I}$. Then if for some $\theta_+ \in \Theta_+$ and $\theta_- \in \Theta_-$, $f : (\mathcal{I}, d_{\theta_+}) \rightarrow (\mathcal{I}, d_{\theta_+})$ and $f : (\mathcal{I}, d_{\theta_-}) \rightarrow (\mathcal{I}, d_{\theta_-})$ are Lipschitz continuous (resp. continuous), then $f : (\mathcal{I}, d_\theta) \rightarrow (\mathcal{I}, d_\theta)$ is Lipschitz continuous (resp. continuous).

Let A be an apartment of \mathcal{I} and $f : \mathcal{I} \rightarrow A$. Suppose that for some $\theta' \in \Theta$, $f : (\mathcal{I}, d_{\theta'}) \rightarrow (A, d_{\theta'})$ is Lipschitz continuous (resp. continuous). Then $f : (\mathcal{I}, d_\theta) \rightarrow (A, d_\theta)$ is Lipschitz continuous (resp. continuous).

Theorem 6.7. *Let $\theta \in \Theta_c$. We equip \mathcal{I} with d_θ . Then:*

- (i) *For all $\theta' \in \Theta_c$, d_θ and $d_{\theta'}$ are equivalent.*
- (ii) *For all $g \in G$, g is Lipschitz-continuous.*
- (iii) *All retraction of \mathcal{I} centered at a sector-germ is Lipschitz continuous.*
- (iv) *The topology induced on each apartment is the affine topology.*
- (v) *There exists $\delta > 0$ such that for all $x, x' \in \mathcal{I}_0$ such that $x \neq x'$, $d(x, x') \geq \delta$.*

Proof. The assertions (i) to (iv) are consequences of Remark 6.6, Theorem 5.7, Corollary 5.9, Theorem 5.14 and Corollary 5.8. Let us prove (v). Let $k \in \mathbb{R}_+^*$ such that $\rho_{+\infty}$ and $\rho_{-\infty}$ are k -Lipschitz continuous. Let $\alpha > 0$ such that for all $x, x' \in Y$, $x \neq x'$ implies $d(x, x') \geq \alpha$. Let $x, x' \in \mathcal{I}_0$ such that $x \neq x'$. By Corollary 4.4 of [Héb16], $\rho_\epsilon(x) \neq \rho_\epsilon(x')$ for some $\epsilon \in \{-\infty, +\infty\}$ and thus $d(x, x') \geq \frac{\alpha}{k}$. \square

Lemma 6.8. *Let $\theta \in \Theta_c$ and $a \in \mathcal{I}$. Let A be an apartment of \mathcal{I} containing a . We fix an origin in A . Let Q_+ and Q_- be opposite sector-germs of A based at 0 and $\rho_+ : \mathcal{I} \xrightarrow{q_+} A$, $\rho_- : \mathcal{I} \xrightarrow{q_-} A$, where q_-, q_+ are the germs of Q_- and Q_+ . Let d be a distance induced by a norm on A . Then there exists $k \in \mathbb{R}_+^*$ such that for all $x \in \mathcal{I}$, $d_\theta(a, x) \leq k(d(a, \rho_-(x)) + d(a, \rho_+(x)))$.*

Proof. One writes $\theta = (\theta_+, \theta_-)$. By Theorem 6.7 (i) and Lemma 5.1, one can suppose that $\theta_- = \theta_{q_-}(d)$, $\theta_+ = \theta_{q_+}(d)$ and $d = d_{\theta|_{A^2}}$.

Let $\nu \in Q$. Let $T_+ = T_\nu : \mathcal{I} \rightarrow A$ and $T_- = T_{-\nu} : \mathcal{I} \rightarrow A$. By Corollary 4.2 and Remark 4.3 of [Héb16], $T_+(x), T_-(x) \leq d(h(\rho_-(x)), h(\rho_+(x)))$ with $h : A \rightarrow \mathbb{R}$ a linear function. Therefore, there exists $l \in \mathbb{R}_+^*$ such that $T_+(x), T_-(x) \leq ld(\rho_-(x), \rho_+(x))$ for all $x \in \mathcal{I}$.

Let $i \in \{-, +\}$ and $x \in \mathcal{I}$. Then

$$d_i(x, \rho_+(x)) \leq d_i(x, x + T(x)\nu) + d_i(\rho_+(x) + T_\nu(x)\nu, \rho_+(x)) \leq 2ld(0, \nu)d_i(\rho_-(x), \rho_+(x)),$$

by definition of T_i .

One has,

$$d(a, x) = d_+(a, x) + d_-(a, x) \leq d_-(a, \rho_-(x)) + d_-(\rho_-(x), x) + d_+(a, \rho_+(x)) + d_+(\rho_+(x), x)$$

$$\text{and thus } d(a, x) \leq (2ld(0, \nu) + 1)(d(a, \rho_-(x)) + d(a, \rho_+(x))).$$

\square

Corollary 6.9. *Let $\theta \in \Theta_c$ and let us equip \mathcal{I} with $d = d_\theta$. Then if $X \subset \mathcal{I}$ the following assertions are equivalent:*

- (i) X is bounded
- (ii) for all retraction ρ centered at a sector of \mathcal{I} , $\rho(X)$ is bounded
- (iii) there exist two opposite sectors q_+ and q_- such that if ρ_{q_-} and ρ_{q_+} are retractions centered at q_- and q_+ , $\rho_{q_-}(X)$ and $\rho_{q_+}(X)$ are bounded.

Moreover each bounded subset of \mathcal{I}_0 is finite.

Proof. By Theorem 6.7, (i) implies (ii), and it is clear that (ii) implies (iii). The implication (iii) implies (i) is a consequence of Lemma 6.8. The last assertion is a consequence of (iii) and of Theorem 5.6 of [Héb16]. \square

Corollary 6.10. *The $\rho_+^{-1}(U) \cap \rho_-^{-1}(U)$ such that U is an open set of an apartment A and ρ_+ and ρ_- are retraction onto A centered at opposed sectors of A form a basis of \mathcal{T}_c . In particular \mathcal{T}_c is the initial topology with respect to retractions centered at sector-germs.*

Proof. This is a consequence of Lemma 6.8. \square

7 Contractibility of \mathcal{I}

In this section we prove the contractibility of \mathcal{I} for \mathcal{T}_+ , \mathcal{T}_- and \mathcal{T}_c . By Theorem 5.7, by symmetry of the role of the Tits cone and of its opposite, it suffices to prove that there exists $\theta \in \Theta(+\infty)$ (where $+\infty = \text{germ}_\infty(C_f^v)$) such that (\mathcal{I}, d_θ) is contractible and Remark 6.6 will conclude for the contractibility of $(\mathcal{I}, \mathcal{T}_c)$.

One chooses a basis $(e_i)_{i \in J}$ of \mathbb{A} such that for some $J' \subset J$, $(e_i)_{i \in J \setminus J'}$ is a basis of \mathbb{A}_{in} and $C_f^v = \bigoplus_{i \in J} \mathbb{R}_+^* e_i \oplus \mathbb{A}_{in}$. Let (e_i^*) be the dual basis of (e_i) . For $x \in \mathbb{A}$, one sets $|x| = \sum_{i \in J} |e_i^*(x)|$. Let $d_{\mathbb{A}}$ be the distance on \mathbb{A} induced by $|\cdot|$ and $\theta = \theta_{+\infty}(d_{\mathbb{A}})$. Let $d = d_\theta$. One has $d_{\mathbb{A}} = d_\theta|_{\mathbb{A}^2}$.

One uses the maps $y_\nu : \mathcal{I} \rightarrow \mathbb{A}$ and $T_\nu : \mathcal{I} \rightarrow \mathbb{R}$, for $\nu \in C_f^v$ defined in Subsection 2.7.

Lemma 7.1. *Let $\nu \in C_f^v$ and $\eta = \min_{i \in I} e_i^*(\nu)$. Then T_ν is $\frac{1}{\eta}$ -Lipschitz continuous and $y_\nu : (\mathcal{I}, d) \rightarrow (\mathbb{A}, d_{\mathbb{A}^2})$ is $\frac{1}{\eta}|\nu| + 1$ -Lipschitz continuous.*

Proof. Let $x, x' \in \mathcal{I}$ and $(u, u') \in T(x, x')$ such that $d(x, x') = |u| + |u'|$. Then $x + T_\nu(x)\nu \in \mathbb{A}$ and thus $x' + u' + T_\nu(x)\nu = x + u + T_\nu(x)\nu \in \mathbb{A}$. As $\frac{1}{\eta}|u|\nu - u \in C_f^v$, $x' + (T_\nu(x) + \frac{1}{\eta}|u|)\nu \in \mathbb{A}$. Consequently, $T_\nu(x') \leq T_\nu(x) + \frac{d(x, x')}{\eta}$ and we get the first part of lemma.

One has

$$\begin{aligned} d(y_\nu(x), y_\nu(x')) &= d(\rho_{+\infty}(x) + T_\nu(x)\nu, \rho_{+\infty}(x') + T_\nu(x')\nu) \\ &\leq d(\rho_{+\infty}(x) + T_\nu(x)\nu, \rho_{+\infty}(x') + T_\nu(x)\nu) + d(\rho_{+\infty}(x') + T_\nu(x)\nu, \rho_{+\infty}(x') + T_\nu(x')\nu) \\ &= d(\rho_{+\infty}(x), \rho_{+\infty}(x')) + d(T_\nu(x)\nu, T_\nu(x')\nu), \end{aligned}$$

because d is invariant by translation on \mathbb{A} .

By Lemma 4.9, $d(\rho_{+\infty}(x), \rho_{+\infty}(x')) \leq d(x, x')$. One also has $d(T_\nu(x)\nu, T_\nu(x')\nu) \leq |T_\nu(x) - T_\nu(x')||\nu| \leq \frac{1}{\eta}|\nu|d(x, x')$ and we can conclude. \square

Remark 7.2. One can also prove that the maps $(\nu, x) \mapsto T_\nu(x)$ and $(\nu, x) \mapsto y_\nu(x)$ are locally Lipschitz continuous.

Proposition 7.3. *Let $\nu \in C_f^v$. One defines $\phi_\nu : \begin{cases} \mathcal{I} \times [0, 1] \rightarrow \mathcal{I} \\ (x, t) \mapsto x + \frac{1}{1-t}\nu \text{ if } \frac{1}{1-t} < T_\nu(x) \\ (x, t) \mapsto y_\nu(x) \text{ if } \frac{1}{1-t} \geq T_\nu(x) \end{cases}$*

(where we consider that $\frac{1}{0} = +\infty > t$ for all $t \in \mathbb{R}$). Then ϕ_ν is a strong deformation retract on \mathbb{A} .

Proof. Let $x \in \mathbb{A}$ and $t \in [0, 1]$. Then $T_\nu(x) = 0$ and thus $\phi_\nu(x) = y_\nu(x) = x$. Let $x \in \mathcal{I}$. Then $\phi_\nu(x, 1) = y_\nu(x) \in \mathbb{A}$. It remains to show that ϕ_ν is continuous. Let $(x_n, t_n) \in (\mathcal{I} \times [0, 1])^\mathbb{N}$ be a converging sequence and $(x, t) = \lim(x_n, t_n)$. Suppose for example that $\frac{1}{1-t} < T_\nu(x)$ (the case $\frac{1}{1-t} = T_\nu(x)$ and $\frac{1}{1-t} > T_\nu(x)$ are analogous). Then by Lemma 7.1, $\frac{1}{1-t_n} < T_\nu(x_n)$ for n large enough and thus by continuity of addition (Lemma 4.5), $\phi_\nu((x_n, t_n)) = x_n + \frac{1}{1-t_n}\nu \rightarrow x + \frac{1}{1-t}\nu = \phi_\nu(x, t)$. Therefore, ϕ_ν is continuous, which concludes the proof. \square

Corollary 7.4. *The measure (\mathcal{I}, d) is contractible.*

Proof. Let $\nu \in Q$. One sets $\psi_\nu : \begin{cases} \mathcal{I} \times [0, 1] \rightarrow \mathcal{I} \\ (x, t) \mapsto \phi_\nu(x, 2t) \text{ if } t \leq \frac{1}{2} \\ (x, t) \mapsto 2(1-t)y_\nu(x) \text{ if } t > \frac{1}{2} \end{cases}$. Then ψ_ν is a strong

deformation retract on $\{0\}$, which proves the corollary. \square

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